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Local Exponential Stabilization of a Coupled Burgers' PDE-ODE System†

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Abstract— **This paper concerns the boundary stabilization problem of a coupled system consisting of the Burgers' equation and a linear ordinary differential equation (ODE). The Burgers' equation is a widely considered nonlinear partial differential equation (PDE), partially due to its low order and partially due to its structure analogous to the Navier-Stokes equation which describes fluid dynamics. The controller we employ for stabilizing this nonlinear coupled system was firstly developed from the boundary control problem of the corresponding linearized system, based on an infinite-dimensional backstepping transformation. By construction of a strict Lyapunov functional, the closed-loop nonlinear system with the backstepping controller is proved to be locally exponentially stable.**

Index Terms— **coupled PDE-ODE systems, nonlinear systems, backstepping, boundary control.**

I. INTRODUCTION

Systems modelled by a coupled PDE and ODE can be found in many engineering problems, e.g., in flexible cable of an overhead crane [1], battery management systems ([2], [3], [4], [5]), and automated drilling systems ([6], [7]). The objective of this paper is to stabilize a coupled system of a (viscous) Burgers' PDE and a linear ODE, through one boundary controller. Burgers' equation is usually considered as a simplified form of the one-dimensional Navier-Stokes equation, and it takes the following form [8]

$$u_t(x,t) \;=\; \epsilon u_{xx}(x,t) - u(x,t)u_x(x,t) \qquad (1)$$

The equation contains a nonlinear term and it is the simplest equation for which the solutions can develop shock waves. Using the Hopf-Cole transformation [9], one can change the equation into a linear parabolic equation. Therefore, the solution do not exhibit chaotic features like sensitivity with respect to initial condition.

Stabilization of PDE systems with boundary control was considered as a challenging topic until around two decades ago, when the backstepping technique which is a systematic method for nonlinear ODE control problems was introduced into PDE control design and estimation problems, see [10] for a tutorial overview of this approach. Since then this method has been used to design stabilizing control laws for many PDEs.

Most of the early efforts focus on the backstepping control of linear PDE and linear coupled PDE-ODE systems. For example, the control problem of a linear coupled hyperbolic PDE-ODE system is studied in [11]. While considering the ODE state as a disturbance source of the hyperbolic PDE, it was shown that the designed backstepping control law, acting on the PDE boundary, is able to attenuate the disturbance. Other works include control with Neumann interconnections [12], coupled ODE-Schrodinger equation [13], and adaptive control for a class of PDE-ODE cascade systems with uncertain harmonic disturbances [14]. It is worth mentioning that the backstepping control technique has also been applied to a coupled system of a linearized Burgers' PDE and a linear ODE, see, [15]. In engineering problems, the backstepping method has found several applications. For examples, the backstepping controller is used in [16] and [17] to find an optimal oil rate under gas coning conditions. Furthermore, the backstepping controller has been used for slugging control [18] and lost circulation and kick control in oil well drilling ([19], [20]).

Although most physical systems are nonlinear, only a few results are available for stabilization of nonlinear PDE systems, such as the Korteweg-de Vries equation ([21], [22]), Benjamin-Bona-Mahony equation [23] and Ginzburg-Landau equation [24]. Even less results exist for stabilizing the coupled systems involving PDEs. For example, feedback control design of nonlinear coupled system of two hetero-directional hyperbolic PDEs, called the 2×2 quasilinear hyperbolic system, is presented in [25], which uses the backstepping technique and achieves local stabilization. To the best knowledge of the authors, the only existing result for stabilizing coupled systems of a linear ODE and a nonlinear PDE is [26], which studies the observer design for a coupled system consisting of a linear ODE and a nonlinear PDE.

In this paper, our control design for the nonlinear coupled Burger's PDE-ODE system follows a similar idea as [25]. In particular, the backstepping feedback controller, which exponentially stabilizes the linearized Burger's PDE-ODE system in the sense of the $\mathbb{H}^1$ norm, is proved to locally stabilize the nonlinear Burgers' PDE-ODE system in the sense of the $\mathbb{H}^2$ norm with an exponential decay rate. The rest of this paper begins with a problem formulation in Section II, which is followed by a presentation of some preliminary results from the boundary control problem of the corresponding linearized coupled PDE-ODE system in Section III. The main result is presented in Section IV, with a proof presented in Section V. Finally, conclusions and some possible future works are presented in section VI.

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II. Problem Formulation

Consider the boundary control problem of the following coupled Burgers' PDE-ODE system

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{2}$$
$$u_t(x,t) = \epsilon u_{xx}(x,t) - u(x,t)u_x(x,t) + CX(t) \tag{3}$$
$$u_x(0,t) = 0 \tag{4}$$
$$u(1,t) = U(t) \tag{5}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state and the pair (A, B) is assumed to be stabilizable; $u(x,t) \in \mathbb{R}$ is the PDE state, and $C^{\intercal}$ is a constant vector; $U(t)$ is the scalar input to the entire system.

Due to the general application of this Burgers' equation to the fluid dynamics, the diffusion coefficient ϵ is a nonnegative constant and is typically referred to as viscosity. In this paper, we are only concerned with the viscous case[1], i.e., $\epsilon > 0$, when the open-loop PDE presents a dissipative characteristic.

In the coupled system, the ODE state has a uniform influence on the PDE, as seen from (2); while on the other hand, the PDE boundary state $u(0,t)$ can also be considered as a force acting on the ODE. The block diagram in Figure 1 shows clearly the control structure, especially the bi-directional influences between PDE and ODE.

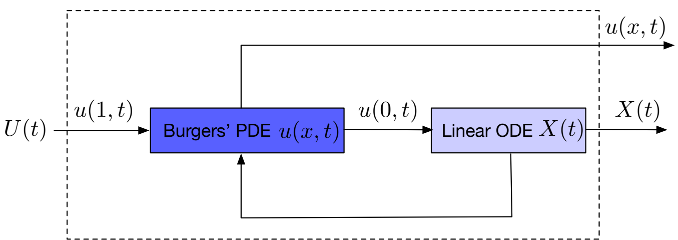

Fig. 1: Control of the coupled Burgers' PDE-ODE system.

III. Preliminaries

A. Definitions and notations

For a vector $Z = (z_i)_{i=\overline{1,n}} \in R^n$, denote its 1-norm as $|Z| = |z_1| + \cdots + |z_n|$. For a real-valued function $f(x,t)$, where $x \in (0,1)$ and $t \in [0,\infty)$, we define $\|f\|_\infty = \sup_{x\in(0,1)} |f|$ and $\|f\|_{\mathbb{L}^1} = \int_0^1 |f| \, \mathrm{d}x$. Furthermore, we define the following spatial norms

$$\|f\|_{\mathbb{L}^2} = \left(\int_0^1 f^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \tag{6}$$

$$\|f\|_{\mathbb{H}^i} = \sum_{k=0}^{i} \left(\int_0^1 \left(\frac{\partial^i f}{\partial x^i}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}, \quad i = 1, \cdots, n \tag{7}$$

[1]When $\epsilon = 0$, the Burgers' equation becomes the inviscid Burgers' equation. It is one of the simplest nonlinear conservative equations, which has a solution in the form of shock waves. However, control of the inviscid Burgers equation is a more challenging problem than the viscous case and is not discussed here.

B. A backstepping control law

Following [15], we would like to introduce the following transformation

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t) \, \mathrm{d}y - \gamma(x)X(t) \tag{8}$$

where

$$q(x,y) = \frac{1}{\epsilon}\int_0^{x-y} \gamma(\xi)B \, \mathrm{d}\xi \tag{9}$$
$$\gamma(x) = \Lambda e^{Dx} E \tag{10}$$

and

$$\Lambda = \begin{pmatrix} K & 0 & \frac{1}{\epsilon}(KA - C) & 0 \end{pmatrix} \tag{11}$$
$$D = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\epsilon^2}BC \\ I & 0 & 0 & 0 \\ 0 & I & 0 & \frac{1}{\epsilon}A \\ 0 & 0 & I & 0 \end{pmatrix} \tag{12}$$
$$E = \begin{pmatrix} I & 0 & 0 & 0 \end{pmatrix}^{\intercal} \tag{13}$$

Here, K is chosen such that $A + BK$ is Hurwitz. It is worth noting that $q(x,y)$ and $\gamma(x)$ satisfy the following coupled PDE system

$$q_{xx}(x,y) = q_{yy}(x,y) \tag{14}$$
$$q(x,x) = 0 \tag{15}$$
$$q_y(x,0) = -\frac{1}{\epsilon}\gamma(x)B \tag{16}$$
$$\epsilon\gamma''(x) = \gamma(x)A + C\int_0^x q(x,y) \, \mathrm{d}y - C \tag{17}$$
$$\gamma(0) = K \tag{18}$$
$$\gamma'(0) = 0 \tag{19}$$

where $(x,y) \in \mathcal{T} = \{(x,y) : 0 \leq y \leq x \leq 1\}$. The transformation (8) is invertible and is given by

$$u(x,t) = w(x,t) + \int_0^x p(x,y)w(y,t) \, \mathrm{d}y + \kappa(x)X(t) \tag{20}$$

where

$$p(x,y) = \frac{1}{\epsilon}\int_0^{x-y} \kappa(\xi)B \, \mathrm{d}\xi \tag{21}$$
$$\kappa(x) = \left(K - C(A+BK)^{-1}\right)G(x) + C(A+BK)^{-1} \tag{22}$$

and

$$G(x) = \begin{pmatrix} I & 0 \end{pmatrix} e^{Hx} \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{23}$$
$$H = \begin{pmatrix} 0 & \frac{1}{\epsilon}(A+BK) \\ I & 0 \end{pmatrix} \tag{24}$$

Based on the backstepping transformation (8), the following linearized system of (2)-(5) around the zero equilibrium $(u, X)^{\intercal} = (0, 0)^{\intercal}$

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{25}$$
$$u_t(x,t) = \epsilon u_{xx}(x,t) + CX(t) \tag{26}$$
$$u_x(0,t) = 0 \tag{27}$$
$$u(1,t) = U(t) \tag{28}$$

is exponentially stabilized by the boundary control law designed as [15]

$$U(t) = \frac{1}{\epsilon}\int_0^1 \left(\int_0^{1-y} \Lambda e^{D\xi}\,\mathrm{d}\xi\right) EBu(y,t)\,\mathrm{d}y + \Lambda e^{D} EX(t) \tag{29}$$

C. Some inequalities

For $f \in \mathbb{H}^2([0,1])$, the following inequalities hold [25]

$$\|f\|_{\mathbb{L}^1} \le a_1\|f\|_{\mathbb{L}^2} \le a_2\|f\|_\infty \tag{30}$$
$$\|f\|_\infty \le a_3\left(\|f\|_{\mathbb{L}^2} + \|f_x\|_{\mathbb{L}^2}\right) \le a_4\|f\|_{\mathbb{H}^1} \tag{31}$$
$$\|f_x\|_\infty \le a_5\left(\|f_x\|_{\mathbb{L}^2} + \|f_{xx}\|_{\mathbb{L}^2}\right) \le a_6\|f\|_{\mathbb{H}^2} \tag{32}$$

where $a_i,\ i = \overline{1,6}$, are positive constants. Moreover, we define the following functionals

$$\mathcal{K}[f] = f(x,t) - \int_0^x q(x,y)f(y,t)\,\mathrm{d}y \tag{33}$$
$$\mathcal{L}[f] = f(x,t) + \int_0^x p(x,y)f(y,t)\,\mathrm{d}y \tag{34}$$
$$\mathcal{K}_1[f] = -q(x,x)f(x,t) + \int_0^x q_y(x,y)f(y,t)\,\mathrm{d}y \tag{35}$$
$$\mathcal{L}_1[f] = p(x,x)f(x,t) + \int_0^x p_x(x,y)f(y,t)\,\mathrm{d}y \tag{36}$$

Since the kernels in both the direct and inverse transformations are $\mathbb{C}^2(\mathcal{T})$ functionals, they satisfy the following inequalities

$$|\mathcal{K}[f]| \le b_1\left(|f| + \|f\|_{\mathbb{L}^1}\right) \tag{37}$$
$$|\mathcal{L}[f]| \le b_2\left(|f| + \|f\|_{\mathbb{L}^1}\right) \tag{38}$$
$$|\mathcal{K}_1[f]| \le b_3\left(|f| + \|f\|_{\mathbb{L}^1}\right) \tag{39}$$
$$|\mathcal{L}_1[f]| \le b_4\left(|f| + \|f\|_{\mathbb{L}^1}\right) \tag{40}$$

for any $(x,t) \in \mathcal{T}$, where $b_i,\ i = \overline{1,4}$ are positive constants.

IV. Main Result

The idea in this paper is to apply the control law (29) to the nonlinear coupled Burgers' PDE-ODE system (2)-(5). Indeed, the solution of (2) is given by

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(0,\tau)\,\mathrm{d}\tau \tag{41}$$

By substituting (41) into (3) and substituting (29) into (5), respectively, the resulting closed-loop system is given by

$$u_t(x,t) = \epsilon u_{xx}(x,t) - u(x,t)u_x(x,t) + C\left(e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(0,\tau)\,\mathrm{d}\tau\right) \tag{42}$$
$$u_x(0,t) = 0 \tag{43}$$
$$u(1,t) = \frac{1}{\epsilon}\int_0^1\left(\int_0^{1-y}\Lambda e^{D\xi}\,\mathrm{d}\xi\right)EBu(y,t)\,\mathrm{d}y + \Lambda e^{D}EX(t) \tag{44}$$

The following theorem holds.

Theorem 1: Consider the closed-loop system (42)-(44) with initial data $X(0) \in \mathbb{R}^n$ and $u_0(x) \in \mathbb{H}^2([0,1])$ compatible with the control law (29). There exists $\delta > 0$ such that if $|X(0)| + \|u_0\|_{\mathbb{H}^2} \le \delta$, the closed-loop system has a unique classical solution and is exponentially stabilized in the sense of the norm

$$\left\|\left(X(t), \dot{X}(t), u(\cdot,t)\right)\right\|^2 = |X(t)|^2 + |\dot{X}(t)|^2 + \|u(\cdot,t)\|^2_{\mathbb{H}^2} \tag{45}$$

where $|\cdot|$ denotes the Euclidean norm.

V. Proof of Theorem 1

In order to prove Theorem 1, we first prove two lemmas.

A. The transformed system

Lemma 1: The transformation (8) maps the system (2)-(5) into the following system

$$\dot{X}(t) = (A + BK)\,X(t) + Bw(0,t) \tag{46}$$
$$w_t(x,t) = \epsilon w_{xx}(x,t) - F[w(x,t), w_x(x,t), X(t)] \tag{47}$$
$$w_x(0,t) = 0 \tag{48}$$
$$w(1,t) = 0 \tag{49}$$

where

$$F[w, w_x, X] = \mathcal{K}\left[\left(\mathcal{L}[w] + \kappa(x)X\right)\left(w_x + \mathcal{L}_1[w] + \kappa'(x)X\right)\right] \tag{50}$$

Proof: Calculating the first and the second order derivatives of (8) with respect to x and the first order derivative of (8) with respect to t, we have

$$w_t(x,t) - \epsilon w_{xx}(x,t) = -u(x,t)u_x(x,t) + \int_0^x q(x,y)u(y,t)u_y(y,t)\,\mathrm{d}y \tag{51}$$

where (2)-(3) and the kernel equation (14)-(19) are used. In view of (34) and (36), and from the inverse transformation (20), we have

$$u(x,t) = \mathcal{L}[w(x,t)] + \kappa(x)X(t) \tag{52}$$
$$u_x(x,t) = \mathcal{L}_x[w(x,t)] + \kappa'(x)X(t) = w_x(x,t) + \mathcal{L}_1[w(x,t)] + \kappa'(x)X(t) \tag{53}$$

Therefore, from (33), we obtain (47) with definition (50). Furthermore, from (2), (8) and (18), we have

$$\dot{X}(t) = AX(t) + B\left(w(0,t) + KX(t)\right) \tag{54}$$

The boundary condition (48) is derived from (27), (15), and (19), and the boundary condition (49) is derived from (8), (5), and (29). Thus, this completes the proof. ■

B. An estimation of the functional F

Lemma 2: There exists $\delta_0 > 0$ such that, if $\|w\|_\infty \le \delta_0$ the functional F satisfies

$$|F| \le c_1\left(|w| + \|w\|_{\mathbb{L}^2}\right)\left(|w_x| + \|w_x\|_{\mathbb{L}^2}\right) + c_2\left(|w|^2 + \|w\|^2_{\mathbb{L}^2}\right) + c_3\left(|w_x| + \|w_x\|_{\mathbb{L}^2}\right)|X(t)| + c_4|X(t)|^2 \tag{55}$$

where $c_i,\ i = \overline{1,4}$, are positive constants.

Proof: The functional F can be written as

$$\begin{aligned} F[w,w_x,X] &= \mathcal{K}\left[(\mathcal{L}[w]+\kappa(x)X)\left(w_x+\mathcal{L}_1[w]+\kappa'(x)X\right)\right] \\ &= \mathcal{K}\left[\mathcal{L}[w]w_x\right]+\mathcal{K}\left[\mathcal{L}[w]\mathcal{L}_1[w]\right] \\ &\quad+\mathcal{K}\left[\mathcal{L}[w]\kappa'(x)X\right]+\mathcal{K}\left[\kappa(x)Xw_x\right] \\ &\quad+\mathcal{K}\left[\kappa(x)X\mathcal{L}_1[w]\right]+\mathcal{K}\left[\kappa(x)X\kappa'(x)X\right] \end{aligned} \tag{56}$$

The first term, with the help of Cauchy-Schwarz inequality and the fact that the kernel $q(x,y)$ is bounded, can be estimated as follows

$$\begin{aligned} &|\mathcal{K}\left[\mathcal{L}[w]w_x\right]| \\ &\le |\mathcal{L}[w]w_x| + \left|\int_0^x q(x,y)\mathcal{L}[w]w_y\,\mathrm{d}y\right| \\ &\le |\mathcal{L}[w]w_x| + c_5\sqrt{\int_0^x q(x,y)^2\left(|w|+\|w\|_{\mathbb{L}^1}\right)^2\,\mathrm{d}y} \\ &\le c_6\left(|w|+\|w\|_{\mathbb{L}^2}\right)|w_x| + c_7\|w\|_{\mathbb{L}^2}\|w_x\|_{\mathbb{L}^2} \end{aligned} \tag{57}$$

where $c_i,\ i=\overline{5,7}$ denote positive constants and (30) is used in the last line. Similarly, the second term can be estimated as follows

$$\begin{aligned} &|\mathcal{K}\left[\mathcal{L}[w]\mathcal{L}_1[w]\right]| \\ &\le |\mathcal{L}[w]\mathcal{L}_1[w]| + \left|\int_0^x q(x,y)\mathcal{L}[w]\mathcal{L}_1[w]\,\mathrm{d}y\right| \\ &\le |\mathcal{L}[w]\mathcal{L}_1[w]| + c_8\sqrt{\int_0^x q(x,y)^2\left(|w|+\|w\|_{\mathbb{L}^1}\right)^4\,\mathrm{d}y} \\ &\le c_9\left(|w|+\|w\|_{\mathbb{L}^2}\right)^2 + c_{10}\|w\|_{\mathbb{L}^2}^2 \end{aligned} \tag{58}$$

where $c_i,\ i=\overline{8,10}$ are suitable positive constants. Since $\kappa(x)$ and $\kappa'(x)$ are bounded, the third term is estimated as

$$\begin{aligned} &|\mathcal{K}\left[\mathcal{L}[w]\kappa'(x)X\right]| \\ &\le |\mathcal{L}[w]\kappa'(x)X| + \left|\int_0^x q(x,y)\mathcal{L}[w]\kappa'(y)X\,\mathrm{d}y\right| \\ &\le |\mathcal{L}[w]\kappa'(x)X| + c_{11}|X|\sqrt{\int_0^x q(x,y)^2\left(|w|+\|w\|_{\mathbb{L}^1}\right)^2\,\mathrm{d}y} \\ &\le c_{12}\left(|w|+\|w\|_{\mathbb{L}^2}\right)|X| + c_{13}|X|\|w\|_{\mathbb{L}^2} \end{aligned} \tag{59}$$

where $c_i,\ i=\overline{11,13}$ are positive constants. The forth term can be estimated as

$$\begin{aligned} &|\mathcal{K}\left[\kappa(x)Xw_x\right]| \\ &\le |\kappa(x)Xw_x| + \left|\int_0^x q(x,y)\kappa(y)Xw_y\,\mathrm{d}y\right| \\ &\le |\kappa(x)Xw_x| + \sqrt{\int_0^x q(x,y)^2\kappa(y)^2|X|^2\,\mathrm{d}y}\sqrt{\int_0^x w_y^2\,\mathrm{d}y} \\ &\le c_{14}|X||w_x| + c_{15}|X|\|w_x\|_{\mathbb{L}^2} \end{aligned} \tag{60}$$

where $c_i,\ i=\overline{14,15}$ denote positive constants. The fifth term is estimated similarly to the third term. Finally the last term can be estimated as follows

$$\begin{aligned} &|\mathcal{K}\left[\kappa(x)X\kappa'(x)X\right]| \\ &\le |\kappa(x)X\kappa'(x)X| + \left|\int_0^x q(x,y)\kappa(y)X\kappa'(y)X\,\mathrm{d}y\right| \\ &\le |\kappa(x)X\kappa'(x)X| \\ &\quad+\sqrt{\int_0^x q(x,y)^2\kappa(y)^2|X|^2\,\mathrm{d}y}\sqrt{\int_0^x \kappa'(x)^2|X|^2\,\mathrm{d}y} \\ &\le c_{16}|X|^2 \end{aligned} \tag{61}$$

where c_{16} denote a positive constant. Adding all terms completes the proof. ■

C. Stability proof using a strict Lyapunov functional

Using Lemma 1 and Lemma 2, we can now prove Theorem 1. Consider the following Lyapunov functional for the transformed system (46)-(49)

$$V_1(t) = X(t)^\intercal PX(t) + \frac{a}{2}\int_0^1 w^2(x,t)\,\mathrm{d}x \tag{62}$$

where $P=P^\intercal>0$ is the unique solution to the Lyapunov equation

$$P\left(A+BK\right)+\left(A+BK\right)^\intercal P = -Q \tag{63}$$

for some positive definite matrix Q. The parameter a is to be determined later. Computing the time derivative of $V_1(t)$, we have

$$\begin{aligned} \dot V_1(t) &= -X(t)^\intercal QX(t) + 2X(t)^\intercal PBw(0,t) \\ &\quad- a\epsilon\int_0^1 w_x^2(x,t)\,\mathrm{d}x \\ &\quad- a\int_0^1 w(x,t)F[w(x,t),w_x(x,t),X(t)]\,\mathrm{d}x \end{aligned} \tag{64}$$

Using Young's inequality, the second term can be estimated as follow

$$2X(t)^\intercal PBw(0,t) \le \frac{\lambda_{\min}(Q)}{2}|X(t)|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)}w(0,t)^2 \tag{65}$$

The last term can be analyzed as follow

$$\begin{aligned} &\left|\int_0^1 w(x,t)F[w(x,t),w_x(x,t),X(t)]\,\mathrm{d}x\right| \\ &\quad\le d_1\int_0^1 |w(x,t)||F[w(x,t),w_x(x,t),X(t)]|\,\mathrm{d}x \end{aligned} \tag{66}$$

where $d_1>0$. From Lemma 2, there exists a constant $\delta_1>0$, such that for $\|w\|_\infty<\delta_1$, it holds that

$$\begin{aligned} &\int_0^1 |w(x,t)||F[w(x,t),w_x(x,t),X(t)]|\,\mathrm{d}x \\ &\le d_2\left(\|w_x\|_\infty\|w\|_{\mathbb{L}^2}^2 + \|w\|_\infty\|w\|_{\mathbb{L}^2}^2 + \|w_x\|_\infty\|w\|_{\mathbb{L}^2}|X(t)|\right. \\ &\qquad\left.+\|w\|_\infty|X(t)|^2\right) \\ &\le d_3\left(\|w_x\|_\infty\left(\|w\|_{\mathbb{L}^2}^2+|X(t)|^2\right)\right. \\ &\qquad\left.+\|w\|_\infty\left(\|w\|_{\mathbb{L}^2}^2+|X(t)|^2\right)\right) \\ &\le d_4\left(\|w_x\|_\infty V_1(t) + V_1(t)^{\frac{3}{2}}\right) \end{aligned} \tag{67}$$

where $d_i,\ i=\overline{2,4}$ are positive constants. Here, the second line uses (30) and (55), and the third line is obtained using Young's inequality, i.e.,

$$\|w\|_{\mathbb{L}^2}|X(t)| \le \frac{1}{2}\|w\|^2_{\mathbb{L}^2} + \frac{1}{2}|X(t)|^2 \tag{68}$$

The last line is obtained from

$$\|w\|_\infty \le d_5\left(\|w_x\|_\infty + V_1(t)^{\frac{1}{2}}\right) \tag{69}$$

for $d_5>0$, which follows from (30) and (31). Thus, we have

$$\begin{aligned}\dot V_1(t) \le& -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \left(a\epsilon - \frac{8|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2_{\mathbb{L}^2}\\ &+ e_1\left(\|w_x\|_\infty V_1(t) + V_1(t)^{\frac{3}{2}}\right)\end{aligned} \tag{70}$$

for $e_1>0$. Choosing $a > \frac{8|PB|^2}{\epsilon\lambda_{\min}(Q)}$, we have

$$\dot V_1(t) \le -\lambda_1 V_1(t) + C_1\left(\|w_x\|_\infty V_1(t) + V_1(t)^{\frac{3}{2}}\right) \tag{71}$$

where

$$\lambda_1 = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{2} - \frac{4|PB|^2}{a\lambda_{\min}(Q)}\right\} > 0 \tag{72}$$

and $C_1>0$. Note that the right hand side of (71) contains $\|w_x\|_\infty$. According to (32), this term can be bounded by $\|w_{xx}\|_{\mathbb{L}^2}$.

In what follows, we show a relation between $\|w_t\|_{\mathbb{L}^2}$ and $\|w_{xx}\|_{\mathbb{L}^2}$ under the assumption that $\|w\|_\infty$ is small enough. Let us denote $v=w_t$ and $Y=\dot X$. Differentiating (46)-(49) with respect to t, we get

$$\dot Y(t) = (A+BK)\,Y(t) + Bv(0,t) \tag{73}$$

$$\begin{aligned}v_t(x,t) =& \epsilon v_{xx}(x,t) - (\mathcal{L}[w] + \kappa(x)X(t))\,v_x\\ &- F_1[w(x,t), w_x(x,t), v(x,t), X(t), Y(t)]\end{aligned} \tag{74}$$

$$v_x(0,t) = 0 \tag{75}$$

$$v(1,t) = 0 \tag{76}$$

where

$$\begin{aligned}&F_1[w(x,t), w_x(x,t), v(x,t), X(t), Y(t)]\\ &= \mathcal{K}_1[(\mathcal{L}[w] + \kappa(x)X(t))\,v] + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,w_x]\\ &\quad + q(x,0)\,(\mathcal{L}[w(0,t)] + \kappa(0)X(t))\,v(0,t)\\ &\quad + \int_0^x q(x,y)\,(\mathcal{L}_y[w] + \kappa'(y)X(t))\,v\,\mathrm{d}y\\ &\quad + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,\mathcal{L}_1[w]]\\ &\quad + \mathcal{K}[(\mathcal{L}[w] + \kappa(x)X(t))\,\mathcal{L}_1[v]]\\ &\quad + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,\kappa'(x)X(t)]\\ &\quad + \mathcal{K}[(\mathcal{L}[w] + \kappa(x)X(t))\,\kappa'(x)Y(t)]\end{aligned} \tag{77}$$

Similar to the proof of Lemma 2, this functional can be estimated as follows

$$\begin{aligned}|F_1| \le& e_2\,(|w| + \|w\|_{\mathbb{L}^2})\,(|v| + \|v\|_{\mathbb{L}^2})\\ &+ e_3\,(|w_x| + \|w_x\|_{\mathbb{L}^2})\,(|v| + \|v\|_{\mathbb{L}^2})\\ &+ e_4\,(|w(0,t)||v(0,t)| + |X||v(0,t)|)\\ &+ e_5\,(|v| + \|v\|_{\mathbb{L}^2})\,|X| + e_6\,(|w_x| + \|w_x\|_{\mathbb{L}^2})\,|Y|\\ &+ e_7\,(|w| + \|w\|_{\mathbb{L}^2})\,|Y| + e_8|X||Y|\end{aligned} \tag{78}$$

where $e_i,\ i=\overline{2,8}$ are positive constants. Consider the following Lyapunov functional

$$V_2(t) = Y(t)^\intercal P Y(t) + \frac{a}{2}\int_0^1 v^2(x,t)\,\mathrm{d}x \tag{79}$$

Computing its first order partial derivative with respect to time, we have

$$\begin{aligned}&\dot V_2(t) = -Y(t)^\intercal Q Y(t) + 2Y(t)^\intercal PBv(0,t)\\ &- a\epsilon\int_0^1 v_x^2(x,t)\,\mathrm{d}x + \frac{a}{2}\,(\mathcal{L}[w(0,t)] + \kappa(0)X(t))\,v^2(0,t)\\ &+ \frac{a}{2}\int_0^1 (\mathcal{L}_x[w] + \kappa'(x)X(t))\,v^2\,\mathrm{d}x\\ &- a\int_0^1 v(x,t)F_1[w(x,t), w_x(x,t), v(x,y), X(t), Y(t)]\,\mathrm{d}x\end{aligned} \tag{80}$$

The fifth term of the right hand side can be estimated as follows

$$\begin{aligned}&\left|\int_0^1 (\mathcal{L}_x[w] + \kappa'(x)X(t))\,v^2\,\mathrm{d}x\right|\\ &\le \int_0^1 |(w_x + \mathcal{L}_1[w] + \kappa'(x)X(t))|\,|v|^2\,\mathrm{d}x\\ &\le g_1\int_0^1 (|w_x| + |w| + \|w\|_{\mathbb{L}^2} + |X(t)|)\,|v|^2\,\mathrm{d}x\\ &\le g_2\left(\|w_x\|_\infty V_2(t) + V_1(t)^{\frac{1}{2}}V_2(t)\right)\end{aligned} \tag{81}$$

for some positive constants g_1 and g_2. Thus, proceeding similarly with calculation for $\dot V_1(t)$, we have from (78)-(81) that

$$\dot V_2(t) \le -\lambda_2 V_2(t) + C_2\left(\|w_x\|_\infty V_2(t) + V_1(t)^{\frac{1}{2}}V_2(t)\right) \tag{82}$$

Now, from (47), we have

$$v(x,t) = \epsilon w_{xx}(x,t) - F[w(x,t), w_x(x,t), X(t)] \tag{83}$$

Furthermore, we compute the $\mathbb{L}^2$ norm of w_{xx} as follows

$$\begin{aligned}&\|w_{xx}(x,t)\|_{\mathbb{L}^2}\\ &\le \frac{1}{\epsilon}\,(\|v(x,t)\|_{\mathbb{L}^2} + \|F[w(x,t), w_x(x,t), X(t)]\|_{\mathbb{L}^2})\\ &\le \frac{1}{\epsilon}\,(\|v(x,t)\|_{\mathbb{L}^2} + g_3\|w\|_\infty\|w_x(x,t)\|_{\mathbb{L}^2}\\ &\quad + g_4|X(t)|\|w_x(x,t)\|_{\mathbb{L}^2} + g_5|X(t)|^2)\end{aligned} \tag{84}$$

where $g_i,\ i=\overline{3,5}$ are positive constants. If $\|w\|_\infty < \min\left\{\delta_1, \frac{\epsilon}{2g_3}\right\}$ and $|X(t)| < \min\left\{\frac{\epsilon}{2g_4}, \sqrt{\frac{\epsilon}{g_5}}\right\}$, we have

$$\|w_{xx}(x,t)\|_{\mathbb{L}^2} \le g_6\|v(x,t)\|_{\mathbb{L}^2} \tag{85}$$

where $g_6>0$. Therefore, $\|w_x\|_\infty \le g_7 V_2(t)^{\frac{1}{2}}$ for $g_7>0$, where Poincare's and Agmon's inequalities are used. Let us define

$$S(t) = V_1(t) + V_2(t) \tag{86}$$

From (71) and (82), we have

$$\dot S(t) \le -\lambda S(t) + CS(t)^{\frac{3}{2}} \tag{87}$$

for some positive λ and C. Then, for any λ_0 such that $0 < \lambda_0 < \lambda$, we have

$$CS(t)^{\frac{3}{2}} \leq (\lambda - \lambda_0)\, S(t),\ \forall S(t) \leq \sqrt{\frac{\lambda - \lambda_0}{C}} \tag{88}$$

which implies that

$$\dot{S}(t) \leq -\lambda_0 S(t),\ \forall S(t) \leq \sqrt{\frac{\lambda - \lambda_0}{C}} \tag{89}$$

Then for sufficiently small $S(0)$, we have $S(t) \to 0$ exponentially. Since $S(t)$ is equivalent to $|X|^2 + |\dot{X}|^2 + \|w_{xx}\|_{\mathbb{L}^2}$ when $\|w\|_\infty$ and $|X|$ are sufficiently small, and the norm $|X|^2 + |\dot{X}|^2 + \|w_{xx}\|_{\mathbb{L}^2}$ is equivalent to the norm $|X(t)|^2 + |\dot{X}(t)|^2 + \|w(\cdot,t)\|_{\mathbb{H}^2}$, then the (w, X)-system is locally exponentially stable in the sense of the norm

$$|X(t)|^2 + |\dot{X}(t)|^2 + \|w(\cdot,t)\|^2_{\mathbb{H}^2} \tag{90}$$

Furthermore, from the fact that the (w, X) system and (u, X) system are equivalent through the backstepping transformation (8) and its inverse (20), the proof is completed.

VI. Conclusions and Future Works

In this paper, we present a stabilization result of a coupled Burgers' PDE-ODE system, with actuation only at one end of the spatial interval. This full-state feedback controller is developed from the stabilization problem of linearized system, with which $\mathbb{H}^2$ local exponential stability is achieved for the resulting closed-loop (nonlinear) Burgers' PDE-ODE system. Further work includes developing an observer for the coupled Burgers' PDE-ODE system using only one boundary measurement. Output feedback regulation will also be investigated.

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