Research Article

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Asymptotic stability of a Korteweg–de Vries equation with a two-dimensional center manifold

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Abstract: Local asymptotic stability analysis is conducted for an initial-boundary-value problem of a Korteweg–de Vries equation posed on a finite interval $[0, 2\pi\sqrt{7/3}]$. The equation comes with a Dirichlet boundary condition at the left end-point and both the Dirichlet and Neumann homogeneous boundary conditions at the right end-point. It is known that the associated linearized equation around the origin is not asymptotically stable. In this paper, the nonlinear Korteweg–de Vries equation is proved to be locally asymptotically stable around the origin through the center manifold method. In particular, the existence of a two-dimensional local center manifold is presented, which is locally exponentially attractive. Analyzing the Korteweg–de Vries equation restricted on the local center manifold, we obtain a polynomial decay rate of the solution.

Keywords: Korteweg–de Vries equation, nonlinearity, center manifold, asymptotic stability, polynomial decay rate

MSC 2010: 35Q53, 37L10, 93D05, 93D20

1 Introduction

The Korteweg–de Vries (KdV) equation

$$y_t + y_x + yy_x + y_{xxx} = 0$$

was first derived by Boussinesq in [2, (283 bis)] and by Korteweg and de Vries in [14], for describing the propagation of small amplitude long water waves in a uniform channel. This equation is now commonly used to model unidirectional propagation of small amplitude long waves in nonlinear dispersive systems. An excellent reference to help understand both physical motivation and deduction of the KdV equation is the book by Whitham [22].

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Rosier studied in [20] the following nonlinear Neumann boundary control problem for the KdV equation with homogeneous Dirichlet boundary conditions, posed on a finite spatial interval:

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, & t \in (0,\infty), \quad x \in (0,L), \\ y(t,0) = y(t,L) = 0, \quad y_x(t,L) = u(t), & t \in (0,\infty), \\ y(0,x) = y_0(x), \quad x \in (0,L), \end{cases} \tag{1.1}$$

where $L > 0$, the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$, and $u(t) \in \mathbb{R}$ denotes the controller. The equation comes with one boundary condition at the left end-point and two boundary conditions at the right end-point. Rosier first considered the first-order power series expansion of (y,u) around the origin, which gives the following corresponding linearized control system:

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & t \in (0,\infty), \quad x \in (0,L), \\ y(t,0) = y(t,L) = 0, \quad y_x(t,L) = u(t), & t \in (0,\infty), \\ y(0,x) = y_0(x), \quad x \in (0,L). \end{cases} \tag{1.2}$$

By means of multiplier technique and the Hilbert uniqueness method (HUM) [15], Rosier proved that (1.2) is exactly controllable if and only if the length of the spatial domain is not critical, i.e., $L \notin \mathcal{N}$, where $\mathcal{N}$ denotes the following set of critical lengths:

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N}^* \right\}. \tag{1.3}$$

Then, by employing the Banach fixed point theorem, he derived that the nonlinear KdV control system (1.1) is locally exactly controllable around 0 provided that $L \notin \mathcal{N}$. In the cases with critical lengths $L \in \mathcal{N}$, Rosier demonstrated in [20] that there exists a finite dimensional subspace M of $L^2(0,L)$ which is unreachable for the linear system (1.2) when starting from the origin. In [8], Coron and Crépeau treated a critical case of $L = 2k\pi$ (i.e., taking $j = l = k$ in $\mathcal{N}$), where k is a positive integer such that (see [7, Theorem 8.1 and Remark 8.2])

$$(j^2 + l^2 + jl = 3k^2 \text{ and } j, l \in \mathbb{N}^*) \ \Rightarrow\ j = l = k. \tag{1.4}$$

Here, the uncontrollable subspace M for the linear system (1.2) is one-dimensional. However, through a third-order power series expansion of the solution, they showed that the nonlinear term yy_x always allows to "go" in small time into the two directions missed by the linearized control system (1.2), and then, using a fixed point theorem, they deduced the small-time local exact controllability around the origin of the nonlinear control system (1.1). In [4], Cerpa studied the critical case of $L \in \mathcal{N}'$, where

$$\mathcal{N}' := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N}^* \text{ satisfying } j > l \text{ and } j^2 + jl + l^2 \neq m^2 + mn + n^2 \text{ for all } m, n \in \mathbb{N}^* \setminus \{j\} \right\}. \tag{1.5}$$

In this case, the uncontrollable subspace M for the linear system (1.2) is of dimension 2, and Cerpa used a second-order expansion of the solution to the nonlinear control system (1.1) to prove the local exact controllability in large time around the origin of the nonlinear control system (1.1) (the local controllability in small time for this length L is still an open problem). Furthermore, Cerpa and Crépeau considered in [5] the cases when the dimension of M for the linear system (1.2) is higher than 2. They implemented a second-order expansion of the solution to (1.1) for the critical lengths $L \neq 2k\pi$ for any $k \in \mathbb{N}^*$, and implemented an expansion to the third order if $L = 2k\pi$ for some $k \in \mathbb{N}^*$. They showed that the nonlinear term yy_x always allows to "go" into all the directions missed by the linearized control system (1.2) and then proved the local exact controllability in large time around the origin of the nonlinear control system (1.1).

Consider the case when there is no control, i.e., $u = 0$, in (1.1), which gives the following initial-boundary-value KdV problem posed on a finite interval $[0,L]$:

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & t \in (0,\infty), \quad x \in (0,L), \\ y(t,0) = y(t,L) = 0, \quad y_x(t,L) = 0, & t \in (0,\infty), \\ y(0,x) = y_0(x), \quad x \in (0,L), \end{cases} \tag{1.6}$$

where the boundary conditions are homogeneous. For the Lyapunov function

$$E(t) = \frac{1}{2}\|y(t,\cdot)\|^2_{L^2(0,L)} = \frac{1}{2}\int_0^L y^2(t,x)dx, \tag{1.7}$$

we have

$$\dot{E}(t) = -\int_0^L y(y_x + yy_x + y_{xxx})dx = \int_0^L y_x y_{xx} dx = -\frac{1}{2}y_x^2(t,0) \le 0. \tag{1.8}$$

Thus, $0 \in L^2(0,L)$ is stable (see $(\mathcal{P}_1)$ below for the definition of stable) for the KdV equation (1.6). Moreover, it has been proved in [18] that, if $L \notin \mathcal{N}$, then 0 is exponentially stable for the corresponding linearized equation around the origin:

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & t \in (0,\infty), \quad x \in (0,L), \\ y(t,0) = y(t,L) = 0, & y_x(t,L) = 0, \quad t \in (0,\infty), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1.9}$$

which gives the local asymptotic stability around the origin for the nonlinear equation (1.6). However, when $L \in \mathcal{N}$, Rosier pointed out in [20] that equation (1.9) is not asymptotically stable. Inspired by the fact that the nonlinear term yy_x introduces the local exact controllability around the origin into the KdV control system (1.1) with $L \in \mathcal{N}$, we would like to discuss whether the nonlinear term yy_x could introduce local asymptotic stability around the origin for (1.6).

This paper is devoted to investigating the local asymptotic stability of $0 \in L^2(0,L)$ for (1.6) with the critical length

$$L = 2\pi\sqrt{7/3},$$

corresponding to $j = 1$ and $l = 2$ in (1.3). Let us recall that this local asymptotic stability means that the following two properties are satisfied.

$(\mathcal{P}_1)$ Stability: for every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that $\|y_0\|_{L^2(0,L)} < \eta$ implies

$$\|y(t,\cdot)\|_{L^2(0,L)} < \varepsilon \quad \text{for all } t \ge 0.$$

$(\mathcal{P}_2)$ (Local) attractivity: there exists $\varepsilon_0 > 0$ such that $\|y_0\|_{L^2(0,L)} < \varepsilon_0$ implies

$$\lim_{t\to+\infty} \|y(t,\cdot)\|_{L^2(0,L)} = 0.$$

As mentioned above, the stability property $(\mathcal{P}_1)$ is implied by (1.8). Our main concern is thus the local attractivity property $(\mathcal{P}_2)$. We prove the following theorem, where the precise definition of a solution to (1.6) is given in Definition 2.7, and the precise definition of the finite dimensional vector space $M \subset L^2(0,L)$ when $L = 2\pi\sqrt{7/3}$ is given in (2.8).

Theorem 1.1. *Consider the KdV equation* (1.6) *with* $L = 2\pi\sqrt{7/3}$. *There exist* $\delta > 0$, $K > 0$, $\omega > 0$ *and a map* $g : M \to M^\perp$, *where* $M^\perp \subset L^2(0,L)$ *is the orthogonal of* M *for the* L^2*-scalar product, satisfying*

$$g \in C^3(M; M^\perp), \tag{1.10}$$

$$g(0) = 0, \quad g'(0) = 0, \tag{1.11}$$

such that, with

$$G := \{m + g(m);\ m \in M\} \subset L^2(0,L), \tag{1.12}$$

the following three properties hold for every solution y *to* (1.6) *with* $\|y_0\|_{L^2(0,L)} < \delta$:

(i) *Local exponential attractivity of* G:

$$d(y(t,\cdot), G) \le Ke^{-\omega t} d(y_0, G) \quad \text{for all } t > 0, \tag{1.13}$$

where $d(\chi, G)$ *denotes the distance between* $\chi \in L^2(0,L)$ *and* G:

$$d(\chi, G) := \inf\{\|\chi - \psi\|_{L^2(0,L)};\ \psi \in G\}.$$

(ii) *Local invariance of G: If $y_0 \in G$, then $y(t,\cdot) \in G$ for all $t \geq 0$.*
(iii) *If $y_0 \in G$, then there exists $C > 0$ such that*

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \frac{C\|y_0\|_{L^2(0,L)}}{\sqrt{1+t\|y_0\|^2_{L^2(0,L)}}} \quad \text{for all } t \geq 0. \tag{1.14}$$

In particular, $0 \in L^2(0,L)$ is locally asymptotically stable in the sense of the $L^2(0,L)$-norm for (1.6).

Remark 1.2. It can be derived from [9, Theorem 1 and comments] that, for every $L > 0$, there are nonzero stationary solutions with the period of L to the following ordinary differential equation (ODE):

$$\begin{cases} f' + ff' + f''' = 0 & \text{in } [0,L], \\ f(0) = f(L) = 0, \\ f'(L) = 0. \end{cases}$$

That is, besides the origin, there also exist other steady states of the nonlinear KdV equation (1.6). Therefore, $0 \in L^2(0,L)$ is not globally asymptotically stable for (1.6): Property $(\mathcal{P}_2)$ does not hold for arbitrary $\varepsilon_0 > 0$.

Our proof of Theorem 1.1 relies on the center manifold approach. This center manifold is G in Theorem 1.1. Center manifold theory plays an important role in studying dynamic properties of nonlinear systems near "critical situations". The center manifold theorem was first proved for finite dimensional systems by Pliss [19] and Kelley [12], and the readers could refer to [13, 17] for more details of this theory. Analogous results are also established for infinite dimensional systems, such as partial differential equations (PDEs) [1, 3] and functional differential equations [10]. The center manifold method usually leads to a dimension reduction of the original problems. Then, in order to derive stability properties (asymptotic stability or unstability) of the full nonlinear equations, one only needs to analyze the reduced equation (restricted on the center manifold). When dealing with the infinite dimensional problems, this method can be extremely efficient if the center manifold is finite dimensional. Following the results on existence, smoothness and attractivity of a center manifold for evolution equations in [21], Chu, Coron and Shang studied in [6] the local asymptotic stability property of (1.6) with the critical length $L = 2k\pi$ for any positive integer k such that (1.4) holds. They proved the existence of a one-dimensional local center manifold. By analyzing the resulting one-dimensional reduced equation, they obtained the local asymptotic stability of 0 for (1.6). For $L = 2\pi\sqrt{7/3}$, we get, following [6], the existence of a two-dimensional local center manifold. It is predictable that the two-dimensional local center manifold introduces more complexity than the one-dimensional local center manifold case.

The organization of this paper is as follows. In Section 2, some basic properties of the linearized KdV equation (1.9) and the KdV equation (1.6) are given. Then, in Section 3, we recall a theorem on the existence of a local center manifold for the KdV equation (1.6) and analyze the dynamics on the local center manifold. Theorem 1.1 follows from this analysis. In Section 4, we present the conclusion and some possible future works. We end this article with Appendix A containing computations which are important for the study of the dynamics on the center manifold.

2 Preliminaries

2.1 Some properties for the linearized equation of (1.6) around the origin

The origin $y = 0$ is an equilibrium of the initial-boundary-value nonlinear KdV problem (1.6). In this subsection, we derive some properties for the linearized KdV equation (1.9) around the origin of (1.6) posed on the finite interval $[0,L]$, where $L = 2\pi\sqrt{7/3} \in \mathcal{N}'$, for which there exists a unique pair $\{j = 2, l = 1\}$ satisfying (1.5).

Let $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L)$ be the linear operator defined by

$$\mathcal{A}\varphi := -\varphi' - \varphi''',$$

with

$$D(\mathcal{A}) := \{\varphi \in H^3(0,L);\ \varphi(0) = \varphi(L) = \varphi'(L) = 0\} \subset L^2(0,L).$$

Then the linearized equation (1.9) can be written as an evolution equation in $L^2(0,L)$:

$$\frac{dy(t,\cdot)}{dt} = \mathcal{A}y(t,\cdot).$$

The following lemma can be immediately obtained.

Lemma 2.1. *$\mathcal{A}^{-1}$ exists and is compact on $L^2(0,L)$. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only: $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ denotes the set of eigenvalues of $\mathcal{A}$.*

Proof. By calculation, we get

$$\mathcal{A}^{-1}\varphi = \psi \quad \text{for all } \varphi \in L^2(0,L),$$

with

$$\psi := -\frac{1-\cos(x-L)}{1-\cos L}\int_0^L (1-\cos y)\varphi(y)\,dy + \int_x^L (1-\cos(x-y))\varphi(y)\,dy.$$

Hence we get the existence of $\mathcal{A}^{-1}$ and that, by the Sobolev embedding theorem, this operator is compact on $L^2(0,L)$. Therefore, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only. □

The following proposition is proved.

Proposition 2.2 ([20, Proposition 3.1]). *$\mathcal{A}$ generates a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $L^2(0,L)$, that is, for any given initial data $y_0 \in L^2(0,L)$, $S(t)y_0$ is the mild solution of the linearized equation* (1.9)*, and*

$$\|S(t)y_0\|_{L^2(0,L)} \leq \|y_0\|_{L^2(0,L)} \quad \text{for all } t \geq 0.$$

Moreover, $\mathrm{Re}(\lambda) \leq 0$ *for every $\lambda \in \sigma(\mathcal{A})$.*

If $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(\mathcal{A})$, then it follows directly from the ABLP theorem (Arendt–Batty–Lyubich–Phong) [16] that the semigroup $S(t)$ is asymptotically stable on $L^2(0,L)$. Since we only have $\mathrm{Re}(\lambda) \leq 0$ for all $\lambda \in \sigma(\mathcal{A})$, the main concern needs to be put on the eigenvalues on the imaginary axis and their corresponding eigenfunctions. Following the proofs of [6, Lemma 2.6] and [20, Lemma 3.5] yields the next lemma.

Lemma 2.3. *There exists a unique pair of conjugate eigenvalues of $\mathcal{A}$ on the imaginary axis, that is,*

$$\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \left\{\lambda = \pm iq;\ q = \frac{20}{21\sqrt{21}}\right\}.$$

Moreover, the corresponding eigenfunctions of $\mathcal{A}$ with respect to $\lambda = \pm iq$ are

$$\varphi := C(\varphi_1 \mp i\varphi_2),$$

respectively, where C is an arbitrary constant, and φ_1, φ_2 are two nonzero real-valued functions:

$$\varphi_1(x) = \Theta\Big(\cos\Big(\frac{5}{\sqrt{21}}x\Big) - 3\cos\Big(\frac{1}{\sqrt{21}}x\Big) + 2\cos\Big(\frac{4}{\sqrt{21}}x\Big)\Big), \tag{2.1}$$

$$\varphi_2(x) = \Theta\Big(-\sin\Big(\frac{5}{\sqrt{21}}x\Big) - 3\sin\Big(\frac{1}{\sqrt{21}}x\Big) + 2\sin\Big(\frac{4}{\sqrt{21}}x\Big)\Big), \tag{2.2}$$

with

$$\Theta := \frac{1}{\sqrt{14\pi}}\sqrt[4]{3/7}. \tag{2.3}$$

Remark 2.4. The equations satisfied by φ_1 and φ_2 are

$$\begin{cases} \varphi_1' + \varphi_1''' = -q\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = q\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0. \end{cases} \tag{2.5}$$

Remark 2.5. We have

$$\int_0^L \varphi_1(x)\varphi_2(x)dx = 0, \tag{2.6}$$

and, with the definition of Θ given in (2.3),

$$\|\varphi_1\|_{L^2(0,L)} = \|\varphi_2\|_{L^2(0,L)} = 1. \tag{2.7}$$

From the results in Lemma 2.1, Proposition 2.2 and Lemma 2.3, we obtain the following corollary.

Corollary 2.6. *$\lambda = \pm i\frac{20}{21\sqrt{21}}$ is the unique eigenvalue pair of $\mathcal{A}$ on the imaginary axis, and all the other eigenvalues of $\mathcal{A}$ have negative real parts which are uniformly bounded away from the imaginary axis, i.e., there exists $r > 0$ such that any of the nonzero eigenvalues of $\mathcal{A}$ has a real part which is less than $-r$.*

Let us define

$$M := \operatorname{span}\{\varphi_1, \varphi_2\} = \{m_1\varphi_1 + m_2\varphi_2;\ \mathbf{m} = (m_1, m_2) \in \mathbb{R}^2\} \subset L^2(0, L), \tag{2.8}$$

where φ_1, φ_2 are defined in (2.1), (2.2) and (2.3). Then the following decomposition holds:

$$L^2(0, L) = M \oplus M^\perp,$$

with

$$M^\perp := \Big\{\varphi \in L^2(0, L);\ \int_0^L \varphi(x)\varphi_1(x)dx = 0,\ \int_0^L \varphi(x)\varphi_2(x)dx = 0\Big\}. \tag{2.9}$$

2.2 Some properties of the KdV equation (1.6)

By considering equation (1.6) as a special case (with $f = 0$ and $u = 0$) of [7, (4.6)–(4.8)], we give the following definition for a solution to equation (1.6), which follows from [7, Definition 4.1].

Definition 2.7. Let $T > 0$ and $y_0 \in L^2(0, L)$. A solution to the Cauchy problem (1.6) on $[0, T]$ is a function

$$y \in \mathcal{B} := C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

such that, for every $\tau \in [0, T]$ and for every $\phi \in C^3([0, \tau] \times [0, L])$ satisfying

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0 \quad \text{for all } t \in [0, \tau], \tag{2.10}$$

one has

$$-\int_0^\tau\int_0^L (\phi_t + \phi_x + \phi_{xxx})y\,dx\,dt + \int_0^\tau\int_0^L \phi y y_x\,dx\,dt + \int_0^L y(\tau, x)\phi(\tau, x)dx - \int_0^L y_0(x)\phi(0, x)dx = 0. \tag{2.11}$$

A solution to the Cauchy problem (1.6) on $[0, +\infty)$ is a function

$$y \in C^0([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$$

such that, for every $T > 0$, y restricted to $[0, T] \times (0, L)$ is a solution to (1.6) on $[0, T]$.

Then by considering equation (1.6) as a special case of [8, (A.1)] (with $f = 0$ and $u = 0$), the following two propositions about the existence and uniqueness of the solutions to (1.6) follow directly from [8, Propositions 14 and 15].

Proposition 2.8. *Let $T \in (0, +\infty)$. There exist $\varepsilon = \varepsilon(T) > 0$ and $C = C(T) > 0$ such that, for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2(0,L)} < \varepsilon(T)$, there exists at least one solution y to equation (1.6) on $[0, T]$ which satisfies*

$$\|y\|_{\mathcal{B}} := \max_{t\in[0,T]} \|y(t, \cdot)\|_{L^2(0,L)} + \Big(\int_0^T \|y(t, \cdot)\|^2_{H^1(0,L)} dt \Big)^{1/2} \leq C(T)\|y_0\|_{L^2(0,L)}.$$

Proposition 2.9. *Let $T \in (0, +\infty)$. There exists $C > 0$ such that, for each pair of solutions (y_1, y_2), corresponding to each pair of initial conditions $(y_{10}, y_{20}) \in (L^2(0, L))^2$, to equation (1.6) on $[0, T]$, the following inequalities hold:*

$$\int_0^T \int_0^L (y_{1x}(t, x) - y_{2x}(t, x))^2 dxdt \leq \int_0^L (y_{10}(x) - y_{20}(x))^2 dx \exp(C(1 + \|y_1\|^2_{L^2(0,T;H^1(0,L))} + \|y_2\|^2_{L^2(0,T;H^1(0,L))})),$$

$$\int_0^L (y_1(t, x) - y_2(t, x))^2 dx \leq \int_0^L (y_{10}(x) - y_{20}(x))^2 dx \exp(C(1 + \|y_1\|^2_{L^2(0,T;H^1(0,L))} + \|y_2\|^2_{L^2(0,T;H^1(0,L))})),$$

for all $t \in [0, T]$.

Let us also mention that for every solution y to (1.6) on $[0, T]$ or on $[0, +\infty)$,

$$t \mapsto \|y(t, \cdot)\|^2_{L^2(0,L)} \text{ is a non-increasing function.} \tag{2.12}$$

This can be easily seen by multiplying the first equation of (1.6) with y, integrating on $[0, L]$ and performing integration by parts. One then gets, if y is smooth enough,

$$\frac{d}{dt} \int_0^L y(t, x)^2 dx = -y_x(t, 0)^2,$$

which gives (2.12). The general case follows from a smoothing argument. As a consequence of Proposition 2.8, Proposition 2.9 and (2.12), one sees that (1.6) has one and only one solution defined on $[0, +\infty)$ if $\|y_0\|_{L^2(0,L)} < \varepsilon(1)$.

3 Existence of a center manifold and dynamics on this manifold

Let us start this section by recalling why, as it is classical, the property "$0 \in L^2(0, L)$ is locally asymptotically stable in the sense of the $L^2(0, L)$-norm for (1.6)" stated at the end of Theorem 1.1 is a consequence of the other statements in this theorem. For convenience, let us recall the argument. Let $y_0 \in L^2(0, L)$ be such that $\|y_0\|_{L^2(0,L)} < \delta$ and let y be the solution to (1.6). It suffices to check that

$$y(t, \cdot) \to 0 \quad \text{in } L^2(0, L) \quad \text{as } t \to +\infty. \tag{3.1}$$

By (1.13), (2.12) and the fact that M is of finite dimension, there exists an increasing sequence of positive real numbers $(t_n)_{n\in\mathbb{N}}$ and $z_0 \in L^2(0, L)$ such that

$$t_n \to +\infty \quad \text{as } n \to +\infty,$$

$$y(t_n, \cdot) \to z_0 \quad \text{in } L^2(0, L) \quad \text{as } n \to +\infty, \tag{3.2}$$

$$z_0 \in G \quad \text{and} \quad \|z_0\|_{L^2(0,L)} < \delta. \tag{3.3}$$

Let $z : [0, +\infty) \times (0, L) \to \mathbb{R}$ be the solution to (1.6) satisfying the initial condition $z(0, \cdot) = z_0$. It follows from (1.14) and (3.3) that

$$z(t, \cdot) \to 0 \quad \text{in } L^2(0, L) \quad \text{as } t \to +\infty. \tag{3.4}$$

Let $\eta > 0$. By (3.4), there exists $\tau > 0$ such that

$$\|z(\tau,\cdot)\|_{L^2(0,L)} \le \frac{\eta}{2}. \tag{3.5}$$

By Proposition 2.9 and (3.2),

$$y(t_n+\tau,\cdot) \to z(\tau,\cdot) \quad \text{in } L^2(0,L) \quad \text{as } n \to +\infty. \tag{3.6}$$

By (3.5) and (3.6), there exists $n_0 \in \mathbb{N}$ such that

$$\|y(t_{n_0}+\tau,\cdot)\|_{L^2(0,L)} < \eta,$$

which, together with (2.12), implies that

$$\|y(t,\cdot)\|_{L^2(0,L)} < \eta \quad \text{for all } t \ge t_{n_0}+\tau,$$

which concludes the proof of (3.1).

The remaining parts of this section are organized as follows. We first recall in Section 3.1 a theorem (Theorem 3.1) on the existence of a local center manifold for (1.6). Then in Section 3.2 we analyze the dynamics of (1.6) on this center manifold and deduce Theorem 1.1 from this analysis.

3.1 Existence of a local center manifold

In [6, Theorem 3.1], following [21], the existence of a center manifold for (1.6) was proved for the first critical length, i.e., $L = 2\pi$. The same proof applies for our L (i.e., $L = 2\pi\sqrt{7/3}$) and allows us to get the following theorem.

Theorem 3.1. *There exist $\delta \in (0, \varepsilon(1))$, $K > 0$, $\omega > 0$ and a map $g : M \to M^\perp$ satisfying* (1.10) *and* (1.11) *such that, with G defined by* (1.12)*, the following two properties hold for every solution $y(t,x)$ to* (1.6) *with $\|y_0\|_{L^2(0,L)} < \delta$:*

(i) *Local exponential attractivity of G:*

$$d(y(t,\cdot), G) \le Ke^{-\omega t} d(y_0, G) \quad \textit{for all } t > 0,$$

where $d(\chi, G)$ denotes the distance between $\chi \in L^2(0,L)$ and G:

$$d(\chi, G) := \inf\{\|\chi - \psi\|_{L^2(0,L)};\ \psi \in G\}.$$

(ii) *Local invariance of G: If $y_0 \in G$, then $y(t,\cdot) \in G$ for all $t \ge 0$.*

3.2 Dynamics on the local center manifold

In this section we study the dynamics of (1.6) on G_δ with

$$G_\delta := \{\zeta(x) \in G;\ \|\zeta\|_{L^2(0,L)} < \delta\}.$$

Let

$$\Omega := \{(m_1, m_2) \in \mathbb{R}^2;\ m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2) \in G_\delta\},$$

then Ω is a bounded open subset of $\mathbb{R}^2$ which contains $(0,0) \in \mathbb{R}^2$. Let $\mathbf{m}^0 = (m_1^0, m_2^0) \in \Omega$, and let y be the solution of (1.6) on $[0,+\infty)$ for the initial data $y_0 := m_1^0\varphi_1 + m_2^0\varphi_2 + g(m_1^0\varphi_1 + m_2^0\varphi_2)$. It follows from (2.12) and Theorem 3.1 that $y(t,\cdot) \in G_\delta$ for every $t \in [0,+\infty)$. Hence we can define, for $t \in [0,+\infty)$, $\mathbf{m}(t) = (m_1(t), m_2(t)) \in \Omega$ by requiring that

$$y(t,\cdot) = m_1(t)\varphi_1 + m_2(t)\varphi_2 + g(m_1(t)\varphi_1 + m_2(t)\varphi_2). \tag{3.7}$$

Since $y \in C^0([0,+\infty); L^2(0,L))$, we have $\mathbf{m} \in C^0([0,+\infty); \mathbb{R}^2)$. Let $T > 0$ and $u \in C_0^\infty(0,T)$. We apply (2.11) with $\tau = T$ and $\phi(t,x) := u(t)\varphi_1(x)$ (note that, by (2.4), (2.10) holds). We get

$$-\int_0^T\int_0^L (\dot u(t)\varphi_1(x) + u(t)\varphi_1'(x) + u(t)\varphi_1'''(x))y(t,x)\,dx\,dt + \int_0^T\int_0^L u(t)\varphi_1(x)(yy_x)(t,x)\,dx\,dt = 0. \tag{3.8}$$

From (2.4), (2.9), (3.7) and (3.8), we have

$$-\int_0^T (m_1(t)\dot u(t) - qm_2(t)u(t))\,dt - \frac{1}{2}\int_0^T\int_0^L y^2(t,x)\varphi_1'(x)u(t)\,dx\,dt = 0.$$

Hence, in the sense of distributions on $(0,T)$,

$$\dot m_1 = -qm_2 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_1'\,dx.$$

Similarly, in the sense of distributions on $(0,T)$,

$$\dot m_2 = qm_1 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_2'\,dx.$$

Hence, if we define $F : \Omega \to \mathbb{R}^2$, $\mathbf{m} = (m_1, m_2) \mapsto F(\mathbf{m})$ by

$$F(\mathbf{m}) := \begin{pmatrix} -qm_2 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_1'\,dx \\ qm_1 + \frac{1}{2}\int_0^L (m_1\varphi_1 + m_2\varphi_2 + g(m_1\varphi_1 + m_2\varphi_2))^2\varphi_2'\,dx \end{pmatrix}, \tag{3.9}$$

then

$$\dot{\mathbf{m}} = F(\mathbf{m}). \tag{3.10}$$

Note that, by (1.10) and (3.9), $F \in C^3(\Omega; \mathbb{R}^2)$, which, together with (3.10), implies that

$$\mathbf{m} \in C^4([0,+\infty); \mathbb{R}^2). \tag{3.11}$$

We now estimate g close to $0 \in M$. Let $\psi \in C^3([0,L])$ be such that

$$\psi(0) = \psi(L) = \psi'(0) = 0. \tag{3.12}$$

Using Definition 2.7 with $\phi(t,x) := \psi(x)$, (3.12) and integration by parts, we get

$$-\frac{1}{\tau}\int_0^\tau\int_0^L (\psi' + \psi''')y\,dx\,dt - \frac{1}{2\tau}\int_0^\tau\int_0^L \psi' y^2\,dx\,dt + \int_0^L \frac{1}{\tau}(y(\tau,x) - y_0(x))\psi(x)\,dx = 0. \tag{3.13}$$

Letting $\tau \to 0^+$ in (3.13), and using (3.9), (3.10) and (3.11), we get

$$\begin{aligned} &-\int_0^L (\psi' + \psi''')y_0\,dx - \frac{1}{2}\int_0^L \psi' y_0^2\,dx \\ &\quad + \int_0^L \Big(\dot m_1(0)\varphi_1(x) + \dot m_2(0)\varphi_2(x) + \frac{\partial g}{\partial m_1}(\mathbf{m}^0)\dot m_1(0) + \frac{\partial g}{\partial m_2}(\mathbf{m}^0)\dot m_2(0)\Big)\psi\,dx = 0. \end{aligned} \tag{3.14}$$

We expand g in a neighborhood of $0 \in M$. Using (1.10) and (1.11), there exist

$$a \in M^\perp, \quad b \in M^\perp, \quad c \in M^\perp \tag{3.15}$$

such that

$$g(\alpha\varphi_1+\beta\varphi_2)=\alpha^2 a+\alpha\beta b+\beta^2 c+o(\alpha^2+\beta^2)\quad \text{in } L^2(0,L)\quad \text{as } \alpha^2+\beta^2\to 0, \tag{3.16}$$

$$\frac{\partial g}{\partial m_1}(\alpha\varphi_1+\beta\varphi_2)=2\alpha a+\beta b+o(|\alpha|+|\beta|)\quad \text{in } L^2(0,L)\quad \text{as } |\alpha|+|\beta|\to 0, \tag{3.17}$$

$$\frac{\partial g}{\partial m_2}(\alpha\varphi_1+\beta\varphi_2)=\alpha b+2\beta c+o(|\alpha|+|\beta|)\quad \text{in } L^2(0,L)\quad \text{as } |\alpha|+|\beta|\to 0. \tag{3.18}$$

As usual, by (3.16), we mean that, for every $\varsigma_1>0$, there exists $\varsigma_2>0$ such that

$$\alpha^2+\beta^2\le\varsigma_1 \;\Rightarrow\; \|g(\alpha\varphi_1+\beta\varphi_2)-(\alpha^2 a+\alpha\beta b+\beta^2 c)\|_{L^2(0,L)}\le\varsigma_2(\alpha^2+\beta^2).$$

Similar definitions are used in (3.17), (3.18) and later on. We now expand the left-hand side of (3.14) in terms of m_1^0, m_2^0, $(m_1^0)^2$, $m_1^0 m_2^0$ and $(m_2^0)^2$ as $|m_1^0|+|m_2^0|\to 0$.

For the functions φ_1 and φ_2 defined by (2.1), (2.2) and (2.3), the following equalities can be derived from (2.4), (2.5) and using integrations by parts:

$$\int_0^L \varphi_1(x)\varphi_2'(x)dx=\frac{10}{7\sqrt{21}},\qquad \int_0^L \varphi_2(x)\varphi_1'(x)dx=-\frac{10}{7\sqrt{21}}, \tag{3.19}$$

$$\int_0^L \varphi_1^2(x)\varphi_1'(x)dx=0,\qquad \int_0^L \varphi_2^2(x)\varphi_2'(x)dx=0, \tag{3.20}$$

$$\int_0^L \varphi_1^2(x)\varphi_2'(x)dx=-2c_1,\qquad \int_0^L \varphi_2^2(x)\varphi_1'(x)dx=2\sqrt{3}c_1, \tag{3.21}$$

$$\int_0^L \varphi_1(x)\varphi_2(x)\varphi_1'(x)dx=c_1,\qquad \int_0^L \varphi_1(x)\varphi_2(x)\varphi_2'(x)dx=-\sqrt{3}c_1, \tag{3.22}$$

where the constant c_1 is defined by

$$c_1:=\frac{177147}{392392\pi}\sqrt{1/(2\pi)}\,\sqrt[4]{3/7}.$$

Looking successively at the terms in $(m_1^0)^2$, $m_1^0 m_2^0$ and $(m_2^0)^2$ in (3.14) as $|m_1^0|+|m_2^0|\to 0$, we get, using (3.9), (3.10), (3.16)–(3.18) as well as (3.19)–(3.22),

$$-\int_0^L(\psi_x+\psi_{xxx})a\,dx-\frac12\int_0^L\psi_x\varphi_1^2dx+\int_0^L(-c_1\varphi_2+qb)\psi dx=0, \tag{3.23}$$

$$-\int_0^L(\psi_x+\psi_{xxx})b\,dx-\int_0^L\psi_x\varphi_1\varphi_2dx+\int_0^L(c_1\varphi_1-\sqrt{3}c_1\varphi_2-2qa+2qc)\psi dx=0, \tag{3.24}$$

$$-\int_0^L(\psi_x+\psi_{xxx})c\,dx-\frac12\int_0^L\psi_x\varphi_2^2dx+\int_0^L(\sqrt{3}c_1\varphi_1-qb)\psi dx=0. \tag{3.25}$$

Since (3.23)–(3.25) must hold for every $\psi\in C^3([0,L])$ satisfying (3.12), one gets that a, b and c are of class C^∞ on $[0,L]$ and satisfy

$$\begin{cases} a'+a'''+\varphi_1\varphi_1'-c_1\varphi_2+qb=0,\\ a(0)=a(L)=0,\quad a'(L)=0,\end{cases} \tag{3.26}$$

$$\begin{cases} b'+b'''+\varphi_1\varphi_2'+\varphi_1'\varphi_2+c_1\varphi_1-\sqrt{3}c_1\varphi_2-2qa+2qc=0,\\ b(0)=b(L)=0,\quad b'(L)=0,\end{cases} \tag{3.27}$$

$$\begin{cases} c'+c'''+\varphi_2\varphi_2'+\sqrt{3}c_1\varphi_1-qb=0,\\ c(0)=c(L)=0,\quad c'(L)=0.\end{cases} \tag{3.28}$$

In Appendix A, we derive the unique functions $a : [0, L] \to \mathbb{R}$, $b : [0, L] \to \mathbb{R}$ and $c : [0, L] \to \mathbb{R}$ which are solutions to (3.26), (3.27) and (3.28). From (3.9) and (3.16), and using (3.20)–(3.22), we get that, as $\mathbf{m} \to \mathbf{0} \in \mathbb{R}^2$,

$$F(\mathbf{m}) = \begin{pmatrix} -qm_2 + \sqrt{3}c_1 m_2^2 + c_1 m_1 m_2 + A_1 m_1^3 + B_1 m_1^2 m_2 + C_1 m_1 m_2^2 + D_1 m_2^3 \\ qm_1 - c_1 m_1^2 - \sqrt{3}c_1 m_1 m_2 + A_2 m_1^3 + B_2 m_1^2 m_2 + C_2 m_1 m_2^2 + D_2 m_2^3 \end{pmatrix} + o(|\mathbf{m}|^3), \tag{3.29}$$

with

$$A_1 := \int_0^L a\varphi_1\varphi_1'\,dx, \qquad B_1 := \int_0^L b\varphi_1\varphi_1'\,dx + \int_0^L a\varphi_2\varphi_1'\,dx, \tag{3.30}$$

$$C_1 := \int_0^L c\varphi_1\varphi_1'\,dx + \int_0^L b\varphi_2\varphi_1'\,dx, \quad D_1 := \int_0^L c\varphi_2\varphi_1'\,dx, \tag{3.31}$$

$$A_2 := \int_0^L a\varphi_1\varphi_2'\,dx, \qquad B_2 := \int_0^L b\varphi_1\varphi_2'\,dx + \int_0^L a\varphi_2\varphi_2'\,dx, \tag{3.32}$$

$$C_2 := \int_0^L c\varphi_1\varphi_2'\,dx + \int_0^L b\varphi_2\varphi_2'\,dx, \quad D_2 := \int_0^L c\varphi_2\varphi_2'\,dx. \tag{3.33}$$

Let us now study the local asymptotic stability property of $\mathbf{0} \in \mathbb{R}^2$ for (3.10). We propose two methods for that. The first one is a more direct one, which relies on normal forms for dynamical systems on $\mathbb{R}^2$. The second one, which relies on a Lyapunov approach related to the physics of (1.6), is less direct. However, there is a reasonable hope that this second method can be applied to other critical lengths $L \in \mathcal{N} \setminus 2\pi\mathbb{N}$ for which the dimension of M is larger than 2.

Method 1: Normal form. Let $z := m_1 + im_2 \in \mathbb{C}$. Then

$$m_1 = \frac{z + \overline{z}}{2}, \quad m_2 = \frac{z - \overline{z}}{2i},$$

and it follows from (3.10) and (3.29) that, as $|z| \to 0$,

$$\dot{z} = (iq)z + P_2(z, \overline{z}) + P_3(z, \overline{z}) + o(|z|^3), \tag{3.34}$$

where $P_j(z, \overline{z})$ are polynomials in $z, \overline{z}$ of degree j. To be more precise, we have

$$\begin{aligned} P_2(z, \overline{z}) &:= (\sqrt{3}c_1 m_2^2 + c_1 m_1 m_2) + i(-c_1 m_1^2 - \sqrt{3}c_1 m_1 m_2) \\ &= -\frac{c_1}{2}(\sqrt{3} + i)z^2 + \frac{c_1}{2}(\sqrt{3} - i)z\overline{z} \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} P_3(z, \overline{z}) := {} & (A_1 + iA_2)\Big(\frac{z + \overline{z}}{2}\Big)^3 + (B_1 + iB_2)\Big(\frac{z + \overline{z}}{2}\Big)^2\Big(\frac{z - \overline{z}}{2i}\Big) \\ & + (C_1 + iC_2)\Big(\frac{z + \overline{z}}{2}\Big)\Big(\frac{z - \overline{z}}{2i}\Big)^2 + (D_1 + iD_2)\Big(\frac{z - \overline{z}}{2i}\Big)^3. \end{aligned} \tag{3.36}$$

We can rewrite (3.34) as

$$\dot{z} = (iq)z + \sum_{i+j=2}^{3} \frac{1}{i!j!} g_{ij} z^i \overline{z}^j + o(|z|^3), \tag{3.37}$$

and it is known from [11, pp. 45, 47] that (3.37) has the following Poincaré normal form:

$$\dot{\xi} = (iq)\xi + \rho\xi^2\overline{\xi} + o(|\xi|^3), \tag{3.38}$$

where

$$\rho = \frac{i}{2q}\Big(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\Big) + \frac{g_{21}}{2}. \tag{3.39}$$

According to (3.35) and (3.36), through a simple computation, we have

$$g_{20} = -c_1(\sqrt{3} + i), \quad g_{11} = \frac{c_1}{2}(\sqrt{3} - i), \quad g_{02} = 0, \tag{3.40}$$

$$g_{21} = \frac{1}{4}(3A_1 + i3A_2 - iB_1 + B_2 + C_1 + iC_2 - i3D_1 + 3D_2). \tag{3.41}$$

Using (3.40) and (3.41), the formula of ρ provided by (3.39) gives $\rho = \rho_1 + i\rho_2$, with

$$\rho_1 := \frac{1}{8}(3A_1 + C_1 + B_2 + 3D_2), \quad \rho_2 := -2\frac{c_1^2}{q} + \frac{1}{8}(-B_1 - 3D_1 + 3A_2 + C_2).$$

It follows that we can derive the Poincaré normal form of the reduced equation on the local center manifold (3.38). Moreover, in Cartesian coordinates, (3.38) is

$$\dot{\xi}_1 = -q\xi_2 + (\rho_1\xi_1 - \rho_2\xi_2)(\xi_1^2 + \xi_2^2) + o(|\xi_1|^3 + |\xi_2|^3),$$
$$\dot{\xi}_2 = q\xi_1 + (\rho_1\xi_2 + \rho_2\xi_1)(\xi_1^2 + \xi_2^2) + o(|\xi_1|^3 + |\xi_2|^3),$$

where

$$\xi = \xi_1 + i\xi_2.$$

In polar coordinates, set

$$r = \sqrt{\xi_1^2 + \xi_2^2}, \quad \theta = \arctan\frac{\xi_2}{\xi_1}.$$

We have, as $r \to 0$,

$$\dot{r} = \rho_1 r^3 + o(r^3), \ \dot{\theta} = q + \rho_2 r^2 + o(r^2). \tag{3.42}$$

Now it is clear to see from (3.42) that the origin $\mathbf{0} \in \mathbb{R}^2$ is asymptotically stable for (3.10) if $\rho_1 < 0$ and is not stable if $\rho_1 > 0$. From (2.1)–(2.3), (3.30)–(3.33) and Appendix A, we can obtain all the coefficients A_i, B_i, C_i, D_i $(i = 1, 2)$. Then, using Matlab, it follows that

$$\rho_1 := \frac{1}{8}(3A_1 + C_1 + B_2 + 3D_2) = -0.008766 < 0.$$

A straightforward computation leads to the existence of $C > 0$ such that, at least if $r(0) \in [0, +\infty)$ is small enough, one has for the solution to (3.42),

$$r(t) \le \frac{Cr(0)}{\sqrt{1 + tr(0)^2}} \quad \text{for all } t \in [0, +\infty),$$

which concludes the proof of Theorem 1.1.

Method 2: Lyapunov function. Let us start with a formal motivation. Recall that, by (1.8) and with E defined in (1.7), we have, along the trajectories of (1.6),

$$\dot{E} = -\frac{1}{2}K^2,$$

with $K := y_x(0)$. It is therefore natural to consider the following candidate for a Lyapunov function:

$$V := E - \mu K\dot{K},$$

where $\mu > 0$ is small enough. Indeed, one then gets

$$\dot{V} := -\frac{1}{2}K^2 - \mu(\dot{K})^2 - \mu K\ddot{K},$$

and one may hope to absorb $-\mu K\ddot{K}$ with $-\frac{1}{2}K^2 - \mu(\dot{K})^2$ and get $\dot{V} < 0$ on $G \setminus \{0\}$, at least in a neighborhood of 0.

We follow this strategy together with the approximation of g previously found. For $\mathbf{m} = (m_1, m_2) \in \Omega$, let (see (3.16))

$$\tilde{g} := m_1^2 a + m_1 m_2 b + m_2^2 c \in C^\infty([0, L]), \tag{3.43}$$

$$\tilde{y} := m_1 \varphi_1 + m_2 \varphi_2 + \tilde{g} \in C^\infty([0, L]), \tag{3.44}$$

and

$$\tilde{E} := \frac{1}{2} \int_0^L \tilde{y}^2 \, dx.$$

Then, using (2.4), (2.5) and (3.26)–(3.28) (compare with (3.14)), one gets that, along the trajectories of (3.10), for $\mathbf{m} \in \Omega$ and $\psi \in C^3([0, L])$ satisfying $\psi(0) = \psi(L) = 0$, one has

$$\begin{aligned}
&- \int_0^L (\psi' + \psi''')\tilde{y}\,dx + \psi'(0)(m_1^2 a'(0) + m_1 m_2 b'(0) + m_2^2 c'(0)) \\
&\qquad - \frac{1}{2} \int_0^L \psi_x \tilde{y}^2 \, dx + \int_0^L \Big(\dot{m}_1 \varphi_1 + \dot{m}_2 \varphi_2 + \frac{\partial \tilde{g}}{\partial m_1} \dot{m}_1 + \frac{\partial \tilde{g}}{\partial m_2} \dot{m}_2 \Big) \psi \, dx \\
&= \int_0^L (\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} + \tilde{y}\tilde{y}_x)\psi \, dx \\
&= \int_0^L \Big[m_1^3 (A_1 \varphi_1 + A_2 \varphi_2 - b c_1 + \varphi_1 a' + a \varphi_1') \\
&\qquad + m_1^2 m_2 (B_1 \varphi_1 + B_2 \varphi_2 + 2ac_1 - b\sqrt{3}c_1 - 2cc_1 + \varphi_1 b' + \varphi_2 a' + a\varphi_2' + b\varphi_1') \\
&\qquad + m_1 m_2^2 (C_1 \varphi_1 + C_2 \varphi_2 + 2a\sqrt{3}c_1 + bc_1 - 2c\sqrt{3}c_1 + \varphi_1 c' + \varphi_2 b' + b\varphi_2' + c\varphi_1') \\
&\qquad + m_2^3 (D_1 \varphi_1 + D_2 \varphi_2 + b\sqrt{3}c_1 + \varphi_2 c' + c\varphi_2') + o(|\mathbf{m}|^3) \Big] \psi \, dx \quad \text{as } |\mathbf{m}| \to 0.
\end{aligned} \tag{3.45}$$

Then, using (3.45) with $\psi := \tilde{y}$ (which, by (2.4), (2.5), (3.26)–(3.28), (3.43) and (3.44), satisfies $\psi(0) = \psi(L) = 0$), along the trajectories of (3.10), we have from (2.6), (2.7), (3.15) and (3.29)–(3.33) that the right-hand side of (3.45) is $o(|\mathbf{m}|^4)$, and

$$\dot{\tilde{E}} = -\frac{1}{2}\tilde{K}^2 + o(|\mathbf{m}|^4) \quad \text{as } |\mathbf{m}| \to 0,$$

with $\tilde{K} : \Omega \to \mathbb{R}$ defined by

$$\tilde{K} := a'(0) m_1^2 + b'(0) m_1 m_2 + c'(0) m_2^2. \tag{3.46}$$

Let us emphasize that, even if "along the trajectories of (3.10)" might be misleading, $\dot{\tilde{E}}$ is just a function of $\mathbf{m} \in \Omega$. It is the same for $\dot{\tilde{V}}$, $\dot{\tilde{K}}$, $\ddot{\tilde{K}}$ which appear below. Using (1.11) and (3.9), we have, along the trajectories of (3.10),

$$\dot{\tilde{K}} = qb'(0) m_1^2 + 2q(c'(0) - a'(0)) m_1 m_2 - qb'(0) m_2^2 + o(|\mathbf{m}|^2). \tag{3.47}$$

Using (3.9), we get the existence of $C > 0$ such that, along the trajectories of (3.10),

$$|\ddot{\tilde{K}}| \leqslant C |\mathbf{m}|^2 \quad \text{for all } \mathbf{m} \in \Omega.$$

We can now define our Lyapunov function $\tilde{V}$. Let $\mu \in (0, \frac{1}{4}]$. Let $\tilde{V} : \Omega \to \mathbb{R}$ be defined by

$$\tilde{V} := \tilde{E} - \mu \tilde{K} \dot{\tilde{K}}. \tag{3.48}$$

From (3.48), we have the existence of $\eta_0 > 0$ such that, for every $\mathbf{m} \in \mathbb{R}^2$ satisfying $|\mathbf{m}| < \eta_0$ and along the trajectories of (3.10),

$$\begin{aligned}
\dot{\tilde{V}} &= -\frac{1}{2}\tilde{K}^2 - \mu(\dot{\tilde{K}})^2 - \mu \tilde{K} \ddot{\tilde{K}} + o(|\mathbf{m}|^4) \\
&\leq -\frac{1}{4}\tilde{K}^2 - \mu(\dot{\tilde{K}})^2 + \mu^2 (\ddot{\tilde{K}})^2 + o(|\mathbf{m}|^4)
\end{aligned}$$

$$\leq -\frac{1}{4}\tilde{K}^2 - \mu(\dot{\tilde{K}})^2 + 2\mu^2 C^2|\mathbf{m}|^4$$
$$\leq -\mu(\tilde{K}^2 + (\dot{\tilde{K}})^2 - 2\mu C^2|\mathbf{m}|^4). \tag{3.49}$$

Let us assume for the moment that, for every $\mathbf{m} = (m_1, m_2) \in \mathbb{R}^2$,

$$\begin{cases} a'(0)m_1^2 + b'(0)m_1m_2 + c'(0)m_2^2 = 0, \\ qb'(0)m_1^2 + 2q(c'(0) - a'(0))m_1m_2 - qb'(0)m_2^2 = 0 \end{cases} \Rightarrow \mathbf{m} = \mathbf{0}. \tag{3.50}$$

Then, by homogeneity, there exists $\eta_1 > 0$ such that

$$(a'(0)m_1^2 + b'(0)m_1m_2 + c'(0)m_2^2)^2 + (qb'(0)m_1^2 + 2q(c'(0) - a'(0))m_1m_2 - qb'(0)m_2^2)^2 \geq 2\eta_1|\mathbf{m}|^4 \tag{3.51}$$

for all $\mathbf{m} = (m_1, m_2) \in \mathbb{R}^2$. From (3.46), (3.47) and (3.51), we get the existence of $\eta_2 > 0$ satisfying

$$\tilde{K}^2 + (\dot{\tilde{K}})^2 \geq \eta_1|\mathbf{m}|^4 \quad \text{for all } \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_2. \tag{3.52}$$

From (3.49) and (3.52), we get the existence of $\eta_3 > 0$ such that, for every $\mu \in (0, \eta_3)$,

$$\dot{\tilde{V}} \leq -\frac{\mu}{2}\eta_1|\mathbf{m}|^4 \quad \text{for all } \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_3. \tag{3.53}$$

Moreover, straightforward estimates show that there exists $\eta_4 > 0$ such that, for every $\mu \in (0, \eta_4)$,

$$\eta_4|\mathbf{m}|^2 \leq \tilde{V} \leq \frac{1}{\eta_4}|\mathbf{m}|^2 \quad \text{for all } \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_4,$$

which, together with (3.53), proves the existence of $C > 0$ such that, at least if $\mathbf{m}^0 \in \mathbb{R}^2$ is small enough, the solution to (3.10) satisfies

$$|\mathbf{m}(t)| \leq \frac{C|\mathbf{m}^0|}{\sqrt{1 + t|\mathbf{m}^0|^2}} \quad \text{for all } t \geq 0.$$

It only remains to prove (3.50). From Appendix A, one gets that $c'(0) \approx 0.0118 \neq 0$, then (3.50) holds if $m_1 = 0$. Let us now deal with the case $m_1 \neq 0$. If we divide both polynomials in the two equations on the left-hand side of (3.50) by m_1^2, then the two resulting polynomials have a common root if and only if their resultant is zero. This resultant is the determinant of the Sylvester matrix S:

$$S := \begin{pmatrix} c'(0) & b'(0) & a'(0) & 0 \\ 0 & c'(0) & b'(0) & a'(0) \\ -b'(0) & -2(a'(0) - c'(0)) & b'(0) & 0 \\ 0 & -b'(0) & -2(a'(0) - c'(0)) & b'(0) \end{pmatrix}.$$

Straightforward computations show that

$$\det(S) = a'(0)^3[b'(0) + 4c'(0)] + a'(0)^2[-2b'(0)^2 + b'(0)c'(0) - 8c'(0)^2] \tag{3.54}$$
$$+ a'(0)[5b'(0)^2c'(0) + 4c'(0)^3] - b'(0)^2c'(0)^2 - b'(0)^4. \tag{3.55}$$

From (3.55) and Appendix A (see in particular (A.7)–(A.9)), we have

$$\det(S) \approx -0.0197 \neq 0.$$

Hence, the two resulting polynomials do not have a common root. Thus, (3.50) is proved.

Remark 3.2. It follows from our proof of Theorem 1.1 that the decay rate stated in (1.14) is optimal in the following sense: there exists $\varepsilon > 0$ such that, for every $y_0 \in G$ with $\|y_0\|_{L^2(0,L)} \leq \varepsilon$,

$$\|y(t,\cdot)\|_{L^2(0,L)} \geq \frac{\varepsilon\|y_0\|_{L^2(0,L)}}{\sqrt{1 + t\|y_0\|^2_{L^2(0,L)}}}.$$

For the Lyapunov approach, let us point out that, decreasing if necessary $\eta_3 > 0$, one has, for every $\mu \in (0, \eta_3)$,

$$\dot{\tilde{V}} \geq -\frac{1}{\eta_3}|\mathbf{m}|^4 \quad \text{for all } \mathbf{m} \in \mathbb{R}^2 \text{ such that } |\mathbf{m}| < \eta_3.$$

4 Conclusion and future works

In this article, we have proved that for the critical case of $L = 2\pi\sqrt{7/3}$, $0 \in L^2(0, L)$ is locally asymptotically stable for the KdV equation (1.6). First, we recalled that the equation has a two-dimensional local center manifold. Next, through a second-order power series approximation at $0 \in M$ of the function g defining the local center manifold, we derived the local asymptotic stability of $0 \in L^2(0, L)$ on the local center manifold and obtained a polynomial decay rate for the solution to the KdV equation (1.6) on the center manifold.

Since the KdV equation (1.6) also has other (periodic) steady states than the origin (see Remark 1.2), it remains an open and interesting problem to consider the (local) stability property of these steady states for the KdV equation (1.6). Furthermore, it remains to consider all the other critical cases with a two-dimensional (local) center manifold as well as all the last remaining critical cases, i.e., when the equation has a (local) center manifold with a dimension larger than 2.

A On the solution *a*, *b* and *c* to equations (3.26), (3.27) and (3.28)

Set

$$f_+(x) := a(x) + c(x), \quad f_-(x) := a(x) - c(x), \tag{A.1}$$

and

$$\begin{cases} g_+(x) := \varphi_1(x)\varphi_1'(x) + \varphi_2(x)\varphi_2'(x) + \sqrt{3}c_1\varphi_1(x) - c_1\varphi_2(x), \\ g_-(x) := \varphi_1(x)\varphi_1'(x) - \varphi_2(x)\varphi_2'(x) - \sqrt{3}c_1\varphi_1(x) - c_1\varphi_2(x), \\ g(x) := \varphi_1(x)\varphi_2'(x) + \varphi_1'(x)\varphi_2(x) + c_1\varphi_1(x) - \sqrt{3}c_1\varphi_2(x). \end{cases} \tag{A.2}$$

First, adding each equation of (3.28) to the corresponding equation of (3.26), we have the following ODE for $f_+(x)$:

$$\begin{cases} f_+'''(x) + f_+'(x) + g_+(x) = 0, \\ f_+(0) = f_+(L) = 0, \quad f_+'(L) = 0. \end{cases} \tag{A.3}$$

Second, subtracting each equation of (3.28) from the corresponding equation of (3.26), we obtain

$$\begin{cases} 2qb(x) + f_-'(x) + f_-'''(x) + g_-(x) = 0, \\ f_-(0) = f_-(L) = 0, \quad f_-'(L) = 0, \end{cases} \tag{A.4}$$

which gives

$$b(x) = -\frac{1}{2q}(f_-'(x) + f_-'''(x) + g_-(x)). \tag{A.5}$$

Substitute (A.5) into (3.27), then the following ODE for $f_-(x)$ is obtained:

$$\begin{cases} f_-^{(6)}(x) + 2f_-^{(4)}(x) + f_-''(x) + 4q^2 f_-(x) + g_-'(x) + g_-'''(x) - 2qg(x) = 0, \\ f_-(0) = f_-(L) = f_-'(L) = f_-'''(L) = 0, \\ f_-'(0) + f_-'''(0) = 0, \quad f_-''(L) + f_-^{(4)}(L) = 0, \end{cases} \tag{A.6}$$

where the boundary conditions follow from (2.4), (2.5), (3.27), (A.2), (A.4) and (A.5).

Employing the method of undetermined coefficients, we get the following (unique) solution to the nonhomogeneous ODE (A.3):

$$\begin{aligned} f_+(x) = \sum_{l=1}^{3} C_{+l} f_{+l}(x) &+ c_{+11}\cos\Big(\frac{1}{\sqrt{21}}x\Big) + c_{+12}\sin\Big(\frac{1}{\sqrt{21}}x\Big) + c_{+21}\cos\Big(\frac{3}{\sqrt{21}}x\Big) \\ &+ c_{+31}\cos\Big(\frac{4}{\sqrt{21}}x\Big) + c_{+32}\sin\Big(\frac{4}{\sqrt{21}}x\Big) + c_{+41}\cos\Big(\frac{5}{\sqrt{21}}x\Big) \\ &+ c_{+42}\sin\Big(\frac{5}{\sqrt{21}}x\Big) + c_{+51}\cos\Big(\frac{6}{\sqrt{21}}x\Big) + c_{+61}\cos\Big(\frac{9}{\sqrt{21}}x\Big), \end{aligned}$$

where the fundamental solutions $f_{+l}(x)$, $l = 1, 2, 3$, are

$$f_{+1}(x) = 1, \quad f_{+2}(x) = \cos(x), \quad f_{+3}(x) = \sin(x),$$

and the constants are

$$\begin{aligned}
c_{+11} &= \frac{3c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, & c_{+12} &= \frac{-3\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, & d_{21} &= \frac{\Theta^2\frac{18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3},\\
c_{+31} &= \frac{-2c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, & d_{32} &= \frac{2\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, & d_{41} &= \frac{c_1\Theta}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3},\\
c_{+42} &= \frac{\sqrt{3}c_1\Theta}{-(\frac{1}{\sqrt{21}}) + (\frac{1}{\sqrt{21}})^3}, & d_{51} &= \frac{\Theta^2\frac{18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3}, & d_{61} &= \frac{\Theta^2\frac{-18}{\sqrt{21}}}{(\frac{1}{\sqrt{21}}) - (\frac{1}{\sqrt{21}})^3},
\end{aligned}$$

and

$$C_{+l} = \frac{\det(A_{+l})}{\det(A_+)}, \quad l = 1, 2, 3.$$

Here,

$$A_+ = \begin{pmatrix} f_{+1}(0) & f_{+2}(0) & f_{+3}(0)\\ f_{+1}(L) & f_{+2}(L) & f_{+3}(L)\\ f'_{+1}(L) & f'_{+2}(L) & f'_{+3}(L)\end{pmatrix},$$

and each A_{+l} is the matrix formed by replacing the l-th column of A_+ with a column vector $-b_+$, where

$$b_+ = \begin{pmatrix} b_{+1} & b_{+2} & b_{+3}\end{pmatrix}^\top,$$

and

$$\begin{aligned}
b_{+1} &= c_{+11} + c_{+21} + c_{+31} + c_{+41} + c_{+51} + c_{+61},\\
b_{+2} &= c_{+11}\cos\Big(\frac{1}{\sqrt{21}}L\Big) + c_{+12}\sin\Big(\frac{1}{\sqrt{21}}L\Big) + c_{+21}\cos\Big(\frac{3}{\sqrt{21}}L\Big)\\
&\quad + c_{+31}\cos\Big(\frac{4}{\sqrt{21}}L\Big) + c_{+32}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + c_{+41}\cos\Big(\frac{5}{\sqrt{21}}L\Big)\\
&\quad + c_{+42}\sin\Big(\frac{5}{\sqrt{21}}L\Big) + c_{+51}\cos\Big(\frac{6}{\sqrt{21}}L\Big) + c_{+61}\cos\Big(\frac{9}{\sqrt{21}}L\Big),\\
b_{+3} &= -\frac{1}{\sqrt{21}}c_{+11}\sin\Big(\frac{1}{\sqrt{21}}L\Big) + \frac{1}{\sqrt{21}}c_{+12}\cos\Big(\frac{1}{\sqrt{21}}L\Big) - \frac{3}{\sqrt{21}}c_{+21}\sin\Big(\frac{3}{\sqrt{21}}L\Big)\\
&\quad - \frac{4}{\sqrt{21}}c_{+31}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + \frac{4}{\sqrt{21}}c_{+32}\cos\Big(\frac{4}{\sqrt{21}}L\Big) - \frac{5}{\sqrt{21}}c_{+41}\sin\Big(\frac{5}{\sqrt{21}}L\Big)\\
&\quad + \frac{5}{\sqrt{21}}c_{+42}\cos\Big(\frac{5}{\sqrt{21}}L\Big) - \frac{6}{\sqrt{21}}c_{+51}\sin\Big(\frac{6}{\sqrt{21}}L\Big) - \frac{9}{\sqrt{21}}c_{+61}\sin\Big(\frac{9}{\sqrt{21}}L\Big).
\end{aligned}$$

Similarly, the method of undetermined coefficients gives the following (unique) solution to the non-homogeneous ODE system (A.6):

$$\begin{aligned}
f_-(x) &= \sum_{l=1}^{6} C_{-l}f_{-l}(x) + c_{-11}\cos\Big(\frac{1}{\sqrt{21}}x\Big) + c_{-12}\sin\Big(\frac{1}{\sqrt{21}}x\Big) + c_{-21}\cos\Big(\frac{2}{\sqrt{21}}x\Big)\\
&\quad + c_{-31}\cos\Big(\frac{4}{\sqrt{21}}x\Big) + c_{-32}\sin\Big(\frac{4}{\sqrt{21}}x\Big) + c_{-41}\cos\Big(\frac{5}{\sqrt{21}}x\Big)\\
&\quad + c_{-42}\sin\Big(\frac{5}{\sqrt{21}}x\Big) + c_{-51}\cos\Big(\frac{8}{\sqrt{21}}x\Big) + c_{-61}\cos\Big(\frac{10}{\sqrt{21}}x\Big),
\end{aligned}$$

where the fundamental solutions $f_{-l}(x)$, $l = 1, \dots, 6$, are

$$\begin{aligned}
f_{-1}(x) &= e^{\alpha_1 x}\cos(\beta_1 x), & f_{-2}(x) &= e^{\alpha_1 x}\sin(\beta_1 x), & f_{-3}(x) &= e^{-\alpha_1 x}\cos(\beta_1 x),\\
f_{-4}(x) &= e^{-\alpha_1 x}\sin(\beta_1 x), & f_{-5}(x) &= \cos(\beta_2 x), & f_{-6}(x) &= \sin(\beta_2 x),
\end{aligned}$$

with

$$\alpha_1 = \frac{(20+\sqrt{57})^{\frac{1}{3}} - 7(20+\sqrt{57})^{-\frac{1}{3}}}{2\sqrt{7}},$$
$$\beta_1 = \frac{(20+\sqrt{57})^{\frac{1}{3}} + 7(20+\sqrt{57})^{-\frac{1}{3}}}{2\sqrt{21}}, \quad \beta_2 = \frac{(20+\sqrt{57})^{\frac{1}{3}} + 7(20+\sqrt{57})^{-\frac{1}{3}}}{\sqrt{21}},$$

and the constants are

$$c_{-11} = \frac{-3\Theta^2\frac{40}{21^2} + 4q\Theta^2\frac{2}{\sqrt{21}} + 9qc_1\Theta}{(\frac{1}{\sqrt{21}})^6 - 2(\frac{1}{\sqrt{21}})^4 + (\frac{1}{\sqrt{21}})^2 - 4q^2}, \quad c_{-12} = \frac{-9\sqrt{3}qc_1\Theta}{(\frac{1}{\sqrt{21}})^6 - 2(\frac{1}{\sqrt{21}})^4 + (\frac{1}{\sqrt{21}})^2 - 4q^2},$$
$$c_{-21} = \frac{3\Theta^2\frac{18}{21^2} - 4q\Theta^2\frac{3^2}{\sqrt{21}}}{(\frac{2}{\sqrt{21}})^6 - 2(\frac{2}{\sqrt{21}})^4 + (\frac{2}{\sqrt{21}})^2 - 4q^2}, \quad c_{-31} = \frac{3\Theta^2\frac{240}{21^2} - 4q\Theta^2\frac{12}{\sqrt{21}} - 6qc_1\Theta}{(\frac{4}{\sqrt{21}})^6 - 2(\frac{4}{\sqrt{21}})^4 + (\frac{4}{\sqrt{21}})^2 - 4q^2},$$
$$c_{-32} = \frac{6\sqrt{3}qc_1\Theta}{(\frac{4}{\sqrt{21}})^6 - 2(\frac{4}{\sqrt{21}})^4 + (\frac{4}{\sqrt{21}})^2 - 4q^2}, \quad c_{-41} = \frac{-3\Theta^2\frac{600}{21^2} + 4q\Theta^2\frac{30}{\sqrt{21}} - 3qc_1\Theta}{(\frac{5}{\sqrt{21}})^6 - 2(\frac{5}{\sqrt{21}})^4 + (\frac{5}{\sqrt{21}})^2 - 4q^2},$$
$$c_{-42} = \frac{-3\sqrt{3}qc_1\Theta}{(\frac{5}{\sqrt{21}})^6 - 2(\frac{5}{\sqrt{21}})^4 + (\frac{5}{\sqrt{21}})^2 - 4q^2}, \quad c_{-51} = \frac{3\Theta^2\frac{2048}{21^2} - 4q\Theta^2\frac{16}{\sqrt{21}}}{(\frac{8}{\sqrt{21}})^6 - 2(\frac{8}{\sqrt{21}})^4 + (\frac{8}{\sqrt{21}})^2 - 4q^2},$$
$$c_{-61} = \frac{3\Theta^2\frac{1250}{21^2} + 4q\Theta^2\frac{5}{\sqrt{21}}}{(\frac{10}{\sqrt{21}})^6 - 2(\frac{10}{\sqrt{21}})^4 + (\frac{10}{\sqrt{21}})^2 - 4q^2},$$

and

$$C_{-l} = \frac{\det(A_{-l})}{\det(A_-)}, \quad l = 1, \dots, 6.$$

Here, the matrix A_- is defined by

$$A_- = \begin{pmatrix} \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \alpha_{-4} & \alpha_{-5} & \alpha_{-6} \end{pmatrix}$$

with

$$\alpha_{-l} = \begin{pmatrix} f_{-l}(0) & f_{-l}(L) & f'_{-l}(L) & f'_{-l}(0) + f'''_{-l}(0) & f'''_{-l}(L) & f''_{-l}(L) + f^{(4)}_{-l}(L) \end{pmatrix}^{\mathsf{T}}, \quad l = 1, \dots, 6.$$

Each A_{-l} is the matrix formed by replacing the l-th column of A_- with a column vector $-b_-$, where

$$b_- = \begin{pmatrix} b_{-1} & b_{-2} & b_{-3} & b_{-4} & b_{-5} & b_{-6} \end{pmatrix}^{\mathsf{T}}$$

and

$$\begin{aligned}
b_{-1} &= c_{-11} + c_{-21} + c_{-31} + c_{-41} + c_{-51} + c_{-61}, \\
b_{-2} &= c_{-11}\cos\Big(\frac{1}{\sqrt{21}}L\Big) + c_{-12}\sin\Big(\frac{1}{\sqrt{21}}L\Big) + c_{-21}\cos\Big(\frac{2}{\sqrt{21}}L\Big) \\
&\quad + c_{-31}\cos\Big(\frac{4}{\sqrt{21}}L\Big) + c_{-32}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + c_{-41}\cos\Big(\frac{5}{\sqrt{21}}L\Big) \\
&\quad + c_{-42}\sin\Big(\frac{5}{\sqrt{21}}L\Big) + c_{-51}\cos\Big(\frac{8}{\sqrt{21}}L\Big) + c_{-61}\cos\Big(\frac{10}{\sqrt{21}}L\Big), \\
b_{-3} &= -\frac{1}{\sqrt{21}}c_{-11}\sin\Big(\frac{1}{\sqrt{21}}L\Big) + \frac{1}{\sqrt{21}}c_{-12}\cos\Big(\frac{1}{\sqrt{21}}L\Big) - \frac{2}{\sqrt{21}}c_{-21}\sin\Big(\frac{2}{\sqrt{21}}L\Big) \\
&\quad - \frac{4}{\sqrt{21}}c_{-31}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + \frac{4}{\sqrt{21}}c_{-32}\cos\Big(\frac{4}{\sqrt{21}}L\Big) - \frac{5}{\sqrt{21}}c_{-41}\sin\Big(\frac{5}{\sqrt{21}}L\Big) \\
&\quad + \frac{5}{\sqrt{21}}c_{-42}\cos\Big(\frac{5}{\sqrt{21}}L\Big) - \frac{8}{\sqrt{21}}c_{-51}\sin\Big(\frac{8}{\sqrt{21}}L\Big) - \frac{10}{\sqrt{21}}c_{-61}\sin\Big(\frac{10}{\sqrt{21}}L\Big), \\
b_{-4} &= \frac{20}{21\sqrt{21}}c_{-12}\cos\Big(\frac{1}{\sqrt{21}}L\Big) + \frac{20}{21\sqrt{21}}c_{-32}\cos\Big(\frac{4}{\sqrt{21}}L\Big) - \frac{20}{\sqrt{21}}c_{-42}\cos\Big(\frac{5}{\sqrt{21}}L\Big),
\end{aligned}$$

$$\begin{aligned}b_{-5} = &-\frac{20}{21\sqrt{21}}c_{-11}\sin\Big(\frac{1}{\sqrt{21}}L\Big) + \frac{20}{21\sqrt{21}}c_{-12}\cos\Big(\frac{1}{\sqrt{21}}L\Big) - \frac{34}{21\sqrt{21}}c_{-21}\sin\Big(\frac{2}{\sqrt{21}}L\Big)\\ &- \frac{20}{21\sqrt{21}}c_{-31}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + \frac{20}{21\sqrt{21}}c_{-32}\cos\Big(\frac{4}{\sqrt{21}}L\Big) + \frac{20}{21\sqrt{21}}c_{-41}\sin\Big(\frac{5}{\sqrt{21}}L\Big)\\ &- \frac{20}{21\sqrt{21}}c_{-42}\cos\Big(\frac{5}{\sqrt{21}}L\Big) + \frac{344}{21\sqrt{21}}c_{-51}\sin\Big(\frac{8}{\sqrt{21}}L\Big) + \frac{790}{21\sqrt{21}}c_{-61}\sin\Big(\frac{10}{\sqrt{21}}L\Big),\\ b_{-6} = &-\frac{20}{21^2}c_{-11}\cos\Big(\frac{1}{\sqrt{21}}L\Big) - \frac{20}{21^2}c_{-12}\sin\Big(\frac{1}{\sqrt{21}}L\Big) - \frac{68}{21^2}c_{-21}\cos\Big(\frac{2}{\sqrt{21}}L\Big)\\ &- \frac{80}{21^2}c_{-31}\cos\Big(\frac{4}{\sqrt{21}}L\Big) - \frac{80}{21^2}c_{-32}\sin\Big(\frac{4}{\sqrt{21}}L\Big) + \frac{100}{21^2}c_{-41}\cos\Big(\frac{5}{\sqrt{21}}L\Big)\\ &+ \frac{100}{21^2}c_{-42}\sin\Big(\frac{5}{\sqrt{21}}L\Big) + \frac{2752}{21^2}c_{-51}\cos\Big(\frac{8}{\sqrt{21}}L\Big) + \frac{7900}{21^2}c_{-61}\cos\Big(\frac{10}{\sqrt{21}}L\Big).\end{aligned}$$

Therefore, we derive from (A.1) that

$$\begin{aligned}a(x) &= \frac{1}{2}(f_+(x) + f_-(x))\\ &= \frac{1}{2}\bigg[\sum_{l=1}^{3} C_{+l}f_{+l}(x) + \sum_{l=1}^{6} C_{-l}f_{-l}(x)\\ &\qquad + (c_{+11} + c_{-11})\cos\Big(\frac{1}{\sqrt{21}}x\Big) + (c_{+12} + c_{-12})\sin\Big(\frac{1}{\sqrt{21}}x\Big) + c_{-21}\cos\Big(\frac{2}{\sqrt{21}}x\Big)\\ &\qquad + c_{+21}\cos\Big(\frac{3}{\sqrt{21}}x\Big) + (c_{+31} + c_{-31})\cos\Big(\frac{4}{\sqrt{21}}x\Big) + (c_{+32} + c_{-32})\sin\Big(\frac{4}{\sqrt{21}}x\Big)\\ &\qquad + (c_{+41} + c_{-41})\cos\Big(\frac{5}{\sqrt{21}}x\Big) + (c_{+42} + c_{-42})\sin\Big(\frac{5}{\sqrt{21}}x\Big) + c_{+51}\cos\Big(\frac{6}{\sqrt{21}}x\Big)\\ &\qquad + c_{-51}\cos\Big(\frac{8}{\sqrt{21}}x\Big) + c_{+61}\cos\Big(\frac{9}{\sqrt{21}}x\Big) + c_{-61}\cos\Big(\frac{10}{\sqrt{21}}x\Big)\bigg]\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}c(x) &= \frac{1}{2}(f_+(x) - f_-(x))\\ &= \frac{1}{2}\bigg[\sum_{l=1}^{3} C_{+l}f_{+l}(x) - \sum_{l=1}^{6} C_{-l}f_{-l}(x)\\ &\qquad + (c_{+11} - c_{-11})\cos\Big(\frac{1}{\sqrt{21}}x\Big) + (c_{+12} - c_{-12})\sin\Big(\frac{1}{\sqrt{21}}x\Big) - c_{-21}\cos\Big(\frac{2}{\sqrt{21}}x\Big)\\ &\qquad + c_{+21}\cos\Big(\frac{3}{\sqrt{21}}x\Big) + (c_{+31} - c_{-31})\cos\Big(\frac{4}{\sqrt{21}}x\Big) + (c_{+32} - c_{-32})\sin\Big(\frac{4}{\sqrt{21}}x\Big)\\ &\qquad + (c_{+41} - c_{-41})\cos\Big(\frac{5}{\sqrt{21}}x\Big) + (c_{+42} - c_{-42})\sin\Big(\frac{5}{\sqrt{21}}x\Big) + c_{+51}\cos\Big(\frac{6}{\sqrt{21}}x\Big)\\ &\qquad - c_{-51}\cos\Big(\frac{8}{\sqrt{21}}x\Big) + c_{+61}\cos\Big(\frac{9}{\sqrt{21}}x\Big) - c_{-61}\cos\Big(\frac{10}{\sqrt{21}}x\Big)\bigg].\end{aligned} \tag{A.8}$$

From (A.5), we obtain

$$\begin{aligned}b(x) = &-\frac{1}{2q}(f_-'(x) + f_-'''(x) + g_-(x))\\ = &-\frac{1}{2q}\bigg[\sum_{l=1}^{6} C_{-l}f_{-l}'(x) + \sum_{l=1}^{6} C_{-l}f_{-l}'''(x)\\ &- \Big(\frac{20}{21\sqrt{21}}c_{-11} + \frac{2}{\sqrt{21}}\Theta^2 + 3c_1\Theta\Big)\sin\Big(\frac{1}{\sqrt{21}}x\Big) + \Big(\frac{20}{21\sqrt{21}}c_{-12} + 3\sqrt{3}c_1\Theta\Big)\cos\Big(\frac{1}{\sqrt{21}}x\Big)\\ &- \Big(\frac{34}{21\sqrt{21}}c_{-21} + \frac{9}{\sqrt{21}}\Theta^2\Big)\sin\Big(\frac{2}{\sqrt{21}}x\Big) - \Big(\frac{20}{21\sqrt{21}}c_{-31} + 2c_1\Theta\Big)\sin\Big(\frac{4}{\sqrt{21}}x\Big)\\ &+ \Big(\frac{20}{21\sqrt{21}}c_{-32} - 2\sqrt{3}c_1\Theta\Big)\cos\Big(\frac{4}{\sqrt{21}}x\Big) + \Big(\frac{20}{21\sqrt{21}}c_{-41} + \frac{30}{\sqrt{21}}\Theta^2 + c_1\Theta\Big)\sin\Big(\frac{5}{\sqrt{21}}x\Big)\end{aligned}$$

$$- \left(\frac{20}{21\sqrt{21}} c_{-42} + \sqrt{3} c_1 \Theta \right) \cos\left(\frac{5}{\sqrt{21}} x \right) - \frac{12}{\sqrt{21}} \Theta^2 \sin\left(\frac{6}{\sqrt{21}} x \right)$$

$$+ \left(\frac{8 \times 43}{21\sqrt{21}} c_{-51} - \frac{16}{\sqrt{21}} \Theta^2 \right) \sin\left(\frac{8}{\sqrt{21}} x \right) + \left(\frac{790}{\sqrt{21}} c_{-61} - \frac{5}{\sqrt{21}} \Theta^2 \right) \sin\left(\frac{10}{\sqrt{21}} x \right) \Bigg]. \tag{A.9}$$

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