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Boundary Stabilization of a Coupled Wave-ODE System

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Abstract: In this paper, we consider stabilization of a coupled wave-ODE system by backstepping method through converting the original system into a stable target system. At the same time, we prove that there exists a unique classical solution for gain function equations by iterative method and some technical skills.

Key Words: Coupled wave-ODE system, Stabilization, Backstepping

1 Introduction

In this paper, we consider stabilization of the following coupled wave-ODE system

$$\begin{cases} u_{tt} - u_{xx} = \lambda(x)u_t + \beta(x)u + CX(t), \\ \dot{X}(t) = AX(t) + Bu_x(0,t), \\ u(0,t) = 0, u(1,t) = U(t), X(0) = x_0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where $\lambda(\cdot) \in C^2([0,1]),\ \beta(\cdot) \in C([0,1]), X(t) \in \mathbb{R}^n$ is the ODE state, and the pair (A, B) is assumed to be stabilizable; $u(x,t) \in \mathbb{R}$ is the state of wave equation, C is a known suitable matrix; x_0, $u_0(x)$ and $u_1(x)$ is initial data; $U(t)$ is the boundary control.

Many authors have showed strong interest in considering controllability and stabilization of coupled systems appeared in many practical control systems such as electromagnetic coupling, mechanical coupling and coupled chemical reactions. Controllability of coupled systems has been discussed in [13, 14] and the references therein. As for stabilization, there are many important tools for constructing explicit stabilizing feedback controllers, such as control Lyapunov function, damping, homogeneity, averaging, backstepping and forwarding methods. What we should emphasis is that backstepping method is a very useful method to design controllers for systems, which have been used in kinds of equations, see [4, 5, 7, 8]. As for stabilization of cascaded PDE-ODE system, Krstic discussed them in [1, 2, 3, 9] by backstepping method. As for stabilization of coupled system, the authors discussed stabilization of coupled heat-ODE system by backstepping method in [10]. How to design the boundary feedback controllers of coupled wave-ODE system is also an interesting problem.

In system (1.1), for constant λ, one can eliminate the anti-damping term by introducing the new variable $v(x,t) = e^{-\lambda t}u(x,t)$. However, this idea does not work for spatially varying $\lambda(x)$. The main idea of this paper is to use the transformation

$$w(x,t) = h(x)u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^x s(x,y)u_t(y,t)dy - M(x)X(t) \tag{1.2}$$

and the feedback controller

$$U(t) = \frac{1}{h(1)}\Big\{ \int_0^1 k(1,y)u(y,t)dy + \int_0^1 s(1,y)u_t(y,t)dy + M(1)X(t)\Big\}, \tag{1.3}$$

where the function $h = h(x)$, gain functions $k = k(x,y)$ and $s = s(x,y)$ are to be appropriately chosen later, to convert (1.1) into

$$\begin{cases} w_{tt} - w_{xx} = -d(x)w_t - c(x)w, \\ \dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), \\ w(0,t) = 0, w(1,t) = 0, \\ X(0) = x_0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x) \end{cases} \tag{1.4}$$

with K being chosen such that $A + BK$ is Hurwitz. For positive smooth functions $c(x)$ and $d(x)$, the authors have proved exponential stability for w-equation of (1.4) in [6], by which and the second equation of (1.4), we can obtain exponential stability of $w(x,t)$ and $X(t)$ in (1.4). Then, we can use exponential stability of (1.4) and invertibility of the transformation (1.2) to obtain stability of the closed-loop system (1.1) and (1.3).

The rest of this paper is organized as follows. In Section 2, we will construct the feedback controller and state the main theorem. In Section 3 , we will prove there exists a unique classical solution of gain function equations of $k(x,y),\ s(x,y)$ and $M(x)$.

2 Main theorem and feedback controller design

The transformation $(X(t), u(x,t)) \longmapsto (X(t), w(x,t))$ is postulated in (1.2), where the gain functions $k(x,y)$, $s(x,y) \in \mathbb{R}$ and $M(x) \in \mathbb{R}^n$ are to be determined later. The inverse transformation $(X(t), w(x,t)) \longmapsto (X(t), u(x,t))$ is postulated in a similar way. By

$$w_{tt} - w_{xx} + d(x)w_t + c(x)w = 0,$$

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after some detailed computation and rearranging the terms, we can obtain that

$$
\begin{aligned}
&\Big\{h(x)C-\int_0^x k(x,y)Cdy-\int_0^x s(x,y)CAdy-M(x)A^2\\
&-\int_0^x s(x,y)\lambda(y)Cdy-c(x)M(x)\\
&-d(x)\int_0^x s(x,y)Cdy-d(x)M(x)A+M''(x)\Big\}X(t)\\
&+\int_0^x u(y,t)\Big\{k_{xx}-k_{yy}-(c(x)+\beta(y))k-2\lambda'(y)s_y\\
&-(\lambda(y)\beta(y)+\lambda''(y)+d(x)\beta(y))s-(\lambda(y)+d(x))s_{yy}\Big\}dy\\
&+\int_0^x u_t(y,t)\Big\{s_{xx}-s_{yy}-(\lambda(y)+d(x))k\\
&-[\lambda^2(y)+d(x)\lambda(y)+c(x)+\beta(y)]s\Big\}dy\\
&+\Big\{h(x)\beta(x)+k_y(x,x)+(\lambda(y)s(x,y))_y(x,x)-h''(x)\\
&+\frac{d}{dx}k(x,x)+k_x(x,x)+d(x)s_y(x,x)+c(x)h(x)\Big\}u\\
&+\Big\{\lambda(x)h(x)+s_x(x,x)+\frac{d}{dx}s(x,x)+s_y(x,x)\\
&+d(x)h(x)\Big\}u_t-\Big\{\lambda(x)s(x,x)+2h'(x)+d(x)s(x,x)\Big\}u_x\\
&+\Big\{-\int_0^x s(x,y)CBdy+k(x,0)+s(x,0)\lambda(0)\\
&-M(x)AB-d(x)M(x)B+d(x)s(x,0)\Big\}u_x(0,t)\\
&+\Big\{s(x,0)-M(x)B\Big\}u_{xt}(0,t)=0.
\end{aligned}
\tag{2.1}
$$

Next, we can choose $k(x,y)$, $s(x,y)$ and $M(x)$ to satisfy the following coupled PDEs

$$
\begin{cases}
k_{xx}-k_{yy}-(c(x)+\beta(y))k-(\lambda(y)+d(x))s_{yy}\\
\quad -(\lambda(y)\beta(y)+\lambda''(y)+d(x)\beta(y))s-2\lambda'(y)s_y=0,\\
2\frac{d}{dx}k(x,x)+(\lambda(x)+d(x))\,s_y(x,x)+\lambda'(x)s(x,x)\\
\quad +(c(x)+\beta(x))h(x)-h''(x)=0,\\
k(x,0)=\int_0^x s(x,y)CBdy-\lambda(0)M(x)B+M(x)AB,
\end{cases}
\tag{2.2}
$$

$$
\begin{cases}
s_{xx}-s_{yy}-(\lambda(y)+d(x))k\\
\quad -\big(\lambda^2(y)+d(x)\lambda(y)+c(x)+\beta(y)\big)s=0,\\
2\frac{d}{dx}s(x,x)=-(\lambda(x)+d(x))h(x),\\
2h'(x)=-(\lambda(x)+d(x))s(x,x),\\
s(x,0)=M(x)B
\end{cases}
\tag{2.3}
$$

and

$$
\begin{aligned}
&M''(x)-M(x)(A^2+c(x)I_n+d(x)A)\\
&-\int_0^x s(x,y)\lambda(y)Cdy-\int_0^x s(x,y)CAdy\\
&-\int_0^x k(x,y)Cdy+h(x)C-d(x)\int_0^x s(x,y)Cdy=0.
\end{aligned}
\tag{2.4}
$$

Choosing $h(0)=1$, according to (1.4) and (1.2), we can obtain $M(0)=0$ and $M'(0)=K$. By the second and third equality of (2.3), we can obtain

$$h'(x)h(x)=s(x,x)\frac{d}{dx}s(x,x).$$

Integrating and noticing $h(0)=1$, we get $s(0,0)=0$. Hence

$$\frac{h'(x)}{\sqrt{h^2(x)-1}}=\frac{\lambda(x)+d(x)}{2},$$

which gives that

$$h(x)=\cosh\Big(\int_0^x a(\tau)d\tau\Big),$$

where $a(x):=\frac{\lambda(x)+d(x)}{2}$. So, we have

$$s(x,x)=-\frac{h'(x)}{a(x)}=-\sinh\Big(\int_0^x a(\tau)d\tau\Big).$$

Next, we will compute $k(x,x)$ explicitly. Let us denote

$$f(x):=s_y(x,x).$$

Integrating the second equality of (2.2) and substituting $s(x,x)$ by $-\frac{h'(x)}{a(x)}$, we have

$$
\begin{aligned}
2k(x,x)=h'(x)+\int_0^x\Big\{&-2a(\tau)f(\tau)+\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\\
&-(\beta(\tau)+c(\tau))h(\tau)\Big\}d\tau.
\end{aligned}
\tag{2.5}
$$

Next, we compute $f(x)$. By

$$\frac{d}{dx}s(x,x)=-a(x)h(x),$$

we have

$$s_x(x,x)=-a(x)h(x)-f(x).$$

By the first equality of (2.3) and substituting y with x, we have

$$
\begin{aligned}
&s_{xx}(x,x)-s_{yy}(x,x)\\
&=(s_x(x,x)-s_y(x,x))'\\
&=2a(x)k(x,x)+(\lambda^2(x)+d(x)\lambda(x)+c(x)+\beta(x))s(x,x)\\
&=(-a(x)h(x)-2f(x))'
\end{aligned}
\tag{2.6}
$$

By (2.5) and (2.6), we obtain

$$
\begin{aligned}
&-2f'(x)-a'(x)h(x)-a(x)h'(x)\\
&=a(x)\Big\{h'(x)+\int_0^x\big[-2a(\tau)f(\tau)+\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\\
&-(\beta(\tau)+c(\tau))h(\tau)\big]d\tau\Big\}\\
&+(\lambda^2(x)+d(x)\lambda(x)+c(x)+\beta(x))s(x,x).
\end{aligned}
\tag{2.7}
$$

Simplifying (2.7), we have

$$
\begin{aligned}
&2f'(x)-2a(x)\int_0^x a(\tau)f(\tau)d\tau\\
&=-2a(x)h'(x)-a'(x)h(x)\\
&-a(x)\int_0^x\Big\{\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c(\tau))h(\tau)\Big\}d\tau\\
&+(\lambda^2(x)+d(x)\lambda(x)+c(x)+\beta(x))\sinh\Big(\int_0^x a(\tau)d\tau\Big).
\end{aligned}
\tag{2.8}
$$

By (2.8), $s(x,0)=M(x)B$, $M(0)=0$ and $M'(0)=K$, we obtain that $f(x)$ satisfies

$$\begin{cases} 2f'(x)-2a(x)\int_0^x a(\tau)f(\tau)d\tau = L(x), \\ f(0)=-a(0)-KB, \end{cases} \tag{2.9}$$

where $L(x)$ is defined by

$$\begin{aligned} L(x) := & -2a(x)h'(x)-a'(x)h(x) \\ & -a(x)\int_0^x \Big\{\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c(\tau))h(\tau)\Big\}d\tau \\ & +(\lambda^2(x)+d(x)\lambda(x)+c(x)+\beta(x))\sinh\Big(\int_0^x a(\tau)d\tau\Big). \end{aligned}$$

Differentiating (2.9) on both sides, $f(x)$ should satisfy

$$\begin{cases} 2a(x)f''(x)-2a'(x)f'(x)-2a^3(x)f(x) \\ \qquad\qquad = L'(x)a(x)-L(x)a'(x), \\ f(0)=-a(0)-KB, f'(0)=-\frac{a'(0)}{2}. \end{cases} \tag{2.10}$$

Solving (2.10), we obtain

$$\begin{aligned} f(x) = & (-a(0)-KB)\cosh\Big(\int_0^x a(\tau)d\tau\Big) \\ & +\frac{1}{2}\int_0^x L(y)\cosh\Big(\int_y^x a(\tau)d\tau\Big)dy. \end{aligned} \tag{2.11}$$

Hence

$$\begin{aligned} k(x,x) = & m(x) := \frac{1}{2}h'(x) \\ & +\frac{1}{2}\int_0^x \Big\{-2a(\tau)\Big[(-a(0)-KB)\cosh\Big(\int_0^\tau a(s)ds\Big) \\ & +\frac{1}{2}\int_0^\tau L(y)\cosh\Big(\int_y^\tau a(s)ds\Big)dy\Big] \\ & +\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c(\tau))h(\tau)\Big\}d\tau. \end{aligned} \tag{2.12}$$

Let us define $\rho_i (i=1,2,3,4,5)$ as follows: $\rho_1(x,y)=\lambda(y)+d(x)$, $\rho_2(x,y)=c(x)+\beta(y)$, $\rho_3(x,y)=\lambda(y)\beta(y)+\lambda''(y)+d(x)\beta(y)$, $\rho_4(x,y)=2\lambda'(y)$, $\rho_5(x,y)=\lambda^2(y)+d(x)\lambda(y)+c(x)+\beta(y)$. We can obtain the following coupled system

$$\begin{cases} k_{xx}-k_{yy}=\rho_1 s_{yy}+\rho_2 k+\rho_3 s+\rho_4 s_y, \\ k(x,x)=m(x), \\ k(x,0)=\int_0^x s(x,y)CBdy-\lambda(0)M(x)B+M(x)AB, \end{cases} \tag{2.13}$$

$$\begin{cases} s_{xx}-s_{yy}=\rho_1 k(x,y)+\rho_5 s(x,y), \\ s(x,x)=-sinh\left(\int_0^x a(\tau)d\tau\right), \\ s(x,0)=M(x)B \end{cases} \tag{2.14}$$

and

$$\begin{cases} M''(x)-M(x)(A^2+c(x)I_n+d(x)A) \\ -\int_0^x s(x,y)\lambda(y)Cdy-\int_0^x s(x,y)CAdy \\ -\int_0^x k(x,y)Cdy+h(x)C-d(x)\int_0^x s(x,y)Cdy=0, \\ M(0)=0, \\ M'(0)=K. \end{cases} \tag{2.15}$$

Introducing the space $H_L^1(0,1)$ defined by

$$H_L^1(0,1) := \{w\in H^1(0,1)|w(0)=0\}$$

and endowed with the H^1-norm, denote the domain

$$\mathbb{T} := \{(x,y)\in\mathbb{R}^2|0\le x\le 1,\ \ 0\le y\le x\}.$$

We state the following main theorem.

Theorem 2.1 *Let* $\lambda(\cdot)\in C^2([0,1])$, $\beta(\cdot)\in C([0,1])$, $0<c(\cdot)\in C([0,1])$ *and* $0<d(\cdot)\in C^1([0,1])$, *then there exist functions* $h(\cdot)\in C^2([0,1])$, $M(\cdot)\in C^2([0,1])$, $k(\cdot,\cdot)$ *and* $s(\cdot,\cdot)\in C^2(\mathbb{T})$, *such that for any* $(x_0,u_0(\cdot),u_1(\cdot))\in\mathbb{R}^n\times H_L^1(0,1)\times L^2(0,1)$ *satisfying the compatibility condition*

$$\begin{aligned} u_0(1) = \frac{1}{h(1)}\Big\{ & \int_0^1 k(1,y)u_0(y)dy \\ & +\int_0^1 s(1,y)u_1(y)dy+M(1)X(t)\Big\}, \end{aligned}$$

there exists a unique classical solution for the closed-loop system (1.1) *and* (1.3) *in the space* $C([0,+\infty);H_L^1(0,1))\cap C^1([0,1);L^2(0,1))$. *Moreover,* $\exists\,\omega,\ C>0$ *independent of initial data such that the solution* $u(\cdot,t)$ *and* $X(t)$ *satisfy*

$$\begin{aligned} & \|(u(\cdot,t),u_t(\cdot,t),X(\cdot))\|_{H^1\times L^2\times\|\cdot\|} \\ & \quad \le Ce^{-\omega t}\|(u_0(\cdot),u_1(\cdot),x_0)\|_{H^1\times L^2\times\|\cdot\|}. \end{aligned} \tag{2.16}$$

3 Existence of the gain functions

To prove the existence of solutions for systems(2.13), (2.14) and (2.15), we use the following change of variable

$$\xi=x+y,\ \eta=x-y.$$

Let us define the functions $G=G(\xi,\eta),\ G^s=G^s(\xi,\eta)$ by

$$G(\xi,\eta)=k\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right),\ G^s(\xi,\eta)=s\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right).$$

Set $g_1(\xi):=m(\frac{\xi}{2})$, $g_2(\xi):=-\sinh\left(\int_0^{\frac{\xi}{2}}a(\tau)d\tau\right)$, $f_1(\xi)=\int_0^\xi s(\xi,y)CBdy-\lambda(0)M(\xi)B+M(\xi)AB$, $f_2(\xi)=M(\xi)B$ and $b_i(\xi,\eta)=\rho_i\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)(i=1,2,3,4,5)$. By (2.13) and (2.14), we can obtain the following coupled PDEs

$$\begin{cases} G_{\xi\eta}=b_1(G^s_{\xi\xi}-2G^s_{\xi\eta}+G^s_{\eta\eta})+b_2G \\ \qquad +b_3G^s+b_4(G^s_\xi-G^s_\eta), \\ G(\xi,0)=g_1(\xi), \\ G(\xi,\xi)=f_1(\xi) \end{cases} \tag{3.1}$$

and

$$\begin{cases} G^s_{\xi\eta}=b_1G+b_5G^s, \\ G^s(\xi,0)=g_2(\xi), \\ G^s(\xi,\xi)=f_2(\xi). \end{cases} \tag{3.2}$$

Integrating (3.1), first with respect to η from 0 to η, then with respect to ξ from η to ξ, we obtain

$$\begin{aligned} G(\xi,\eta) = & g_1(\xi)+f_1(\eta)-g_1(\eta)+\int_\eta^\xi\int_0^\eta b_2Gdsd\tau \\ & +\int_\eta^\xi\int_0^\eta b_1(G^s_{\xi\xi}-2G^s_{\xi\eta}+G^s_{\eta\eta})dsd\tau \\ & +\int_\eta^\xi\int_0^\eta \Big\{b_3G^s+b_4(G^s_\xi-G^s_\eta)\Big\}dsd\tau. \end{aligned} \tag{3.3}$$

Similarly, integrating (3.2), first with respect to η from 0 to η, then with respect to ξ from η to ξ, we get

$$G^s(\xi,\eta) = g_2(\xi)+f_2(\eta) - g_2(\eta) + \int_\eta^\xi \int_0^\eta \Big(b_1 G + b_5 G^s\Big) ds d\tau. \tag{3.4}$$

According to (2.15), define

$$F(x) := \int_0^x s(x,y)\lambda(y) C dy + \int_0^x s(x,y) CA dy + \int_0^x k(x,y) C dy + d(x)\int_0^x s(x,y) C dy,$$

we have

$$\begin{cases} M'' = M(A^2 + c(x) I_n + d(x) A) + F(x) - h(x) C, \\ M(0) = 0, \\ M'(0) = K. \end{cases} \tag{3.5}$$

Then

$$\begin{cases} Y'(x) = LY + \overrightarrow{F}(x), \\ Y(0) = Y_0, \end{cases} \tag{3.6}$$

where

$$L = \begin{pmatrix} 0 & I_n \\ (A^2 + c(x) I_n + d(x) A)^T & 0 \end{pmatrix},$$

$\overrightarrow{F}(x) = (0, \tilde{F}(x))^T$, $Y = (M(x), M'(x))^T$, $Y_0 = (0, K)^T$, $\tilde{F}(x) := F(x) - h(x)C$. Set $\Phi(x)$ be fundamental solution matrix of $Y'(x) = LY(x)$, then

$$\begin{aligned} Y(x) &= \Phi(x)\begin{pmatrix} 0 \\ K \end{pmatrix} + \int_0^x \Phi(x)\Phi^{-1}(\tau)\begin{pmatrix} 0 \\ \tilde{F}(\tau) \end{pmatrix} d\tau, \\ &= \begin{pmatrix} \Phi_{11}(x) & \Phi_{12}(x) \\ \Phi_{21}(x) & \Phi_{22}(x) \end{pmatrix}\begin{pmatrix} 0 \\ K \end{pmatrix} \\ &\quad + \int_0^x \begin{pmatrix} \Psi_{11}(x,\tau) & \Psi_{12}(x,\tau) \\ \Psi_{21}(x,\tau) & \Psi_{22}(x,\tau) \end{pmatrix}\begin{pmatrix} 0 \\ \tilde{F}(\tau) \end{pmatrix} d\tau. \end{aligned}$$

where we get the above equality by dividing matrix $\Phi(x)$ and $\Phi(x)\Phi^{-1}(\tau)$ into appropriate block matrix $\Phi_{ij}(x), \Psi_{ij}(x,\tau) (i,j = 1,2)$. Hence, we can obtain

$$M'(x) = \Phi_{22}(x) K + \int_0^x \Psi_{22}(x,\tau)\tilde{F}(\tau) d\tau.$$

Hence

$$M(x) = \int_0^x \Phi_{22}(y) K dy + \int_0^x \int_0^y \Psi_{22}(y,\tau)\tilde{F}(\tau) d\tau dy. \tag{3.7}$$

Substituting $M(x)$ into (3.3) and (3.4), we can obtain

$$\begin{aligned} G(\xi,\eta) &= g_1(\xi) - \lambda(0)\Big\{ \int_0^\eta \Phi_{22}(y) K dy \\ &- \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} B \\ &+ \Big\{ \int_0^\eta \Phi_{22}(y) K dy - \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} AB \\ &- g_1(\eta) + \int_0^\eta s(\eta, y) CB dy \\ &- \lambda(0)\Big\{ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F(\tau) d\tau dy \Big\} B \\ &+ \Big\{ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F(\tau) d\tau dy \Big\} AB + \int_\eta^\xi \int_0^\eta b_2 G ds d\tau \\ &+ \int_\eta^\xi \int_0^\eta b_1 (G^s_{\xi\xi} - 2G^s_{\xi\eta} + G^s_{\eta\eta}) ds d\tau \\ &+ \int_\eta^\xi \int_0^\eta \Big\{ b_3 G^s + b_4 (G^s_\xi - G^s_\eta) \Big\} ds d\tau \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} G^s(\xi,\eta) &= g_2(\xi) - g_2(\eta) \\ &+ \Big\{ \int_0^\eta \Phi_{22}(y) K dy - \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} B \\ &+ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F(\tau) d\tau dy B + \int_\eta^\xi \int_0^\eta (b_1 G + b_5 G^s) ds d\tau. \end{aligned} \tag{3.9}$$

We will use a classical iterative method to prove that system (3.8) and (3.9) has a unique classical solution. We define G^1 and $G^{s,1}$ as

$$\begin{aligned} G^1(\xi,\eta) &:= g_1(\xi) - g_1(\eta) - \lambda(0)\Big\{ \int_0^\eta \Phi_{22}(y) K dy \\ &- \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} B \\ &+ \Big\{ \int_0^\eta \Phi_{22}(y) K dy - \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} AB, \end{aligned}$$

$$\begin{aligned} G^{s,1}(\xi,\eta) &:= g_2(\xi) - g_2(\eta) \\ &+ \Big\{ \int_0^\eta \Phi_{22}(y) K dy - \int_0^\eta \int_0^y \Psi_{22}(y,\tau) h(\tau) C d\tau dy \Big\} B. \end{aligned}$$

We next construct the following recursion for $n = 1, 2, 3, \cdots$

$$\begin{aligned} &G^{n+1}(\xi,\eta) \\ &= \int_0^\eta G^{s,n}(\eta + y, \eta - y) CB dy \\ &- \lambda(0)\Big\{ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F^n(\tau) d\tau dy \Big\} B \\ &+ \Big\{ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F^n(\tau) d\tau dy \Big\} AB + \int_\eta^\xi \int_0^\eta b_2 G^n ds d\tau \\ &+ \int_\eta^\xi \int_0^\eta b_1 (G^{s,n}_{\xi\xi} - 2G^{s,n}_{\xi\eta} + G^{s,n}_{\eta\eta}) ds d\tau \\ &+ \int_\eta^\xi \int_0^\eta \Big\{ b_3 G^{s,n} + b_4 (G^{s,n}_\xi - G^{s,n}_\eta) \Big\} ds d\tau \end{aligned} \tag{3.10}$$

and

$$G^{s,n+1}(\xi,\eta) = \Big\{ \int_0^\eta \int_0^y \Psi_{22}(y,\tau) F^n(\tau) d\tau dy \Big\} B + \int_\eta^\xi \int_0^\eta (b_1 G^n + b_5 G^{s,n}) ds d\tau. \tag{3.11}$$

By the definition of $G^1(\xi,\eta)$ and $G^{s,1}(\xi,\eta)$, we can find large enough number M such that $|G^1(\xi,\eta)|$, $|G^1_\xi(\xi,\eta)|, |G^1_\eta(\xi,\eta)|, |G^{s,1}(\xi,\eta)|, |G^{s,1}_\xi(\xi,\eta)|, |G^{s,1}_\eta(\xi,\eta)|$, $|G^{s,1}_{\xi,\eta}(\xi,\eta)|, |G^{s,1}_{\xi,\xi}(\xi,\eta)|, |G^{s,1}_{\xi,\eta}(\xi,\eta)|$ are smaller than M. Let us now suppose that for some $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|G^n| \le MK^n \frac{(\xi+\eta)^n}{n!},\\
&|G^{s,n}| \le MK^n \frac{(\xi+\eta)^{n+1}}{(n+1)!},\\
&|G^n_\xi|, \quad |G^n_\eta| \le MK^n \frac{(\xi+\eta)^{n-1}}{(n-1)!},\\
&|G^{s,n}_\xi|, \quad |G^{s,n}_\eta| \le MK^n \frac{(\xi+\eta)^n}{n!},\\
&|G^{s,n}_{\xi\xi}|, \ |G^{s,n}_{\xi\eta}|, \quad |G^{s,n}_{\eta\eta}| \le MK^n \frac{(\xi+\eta)^{n-1}}{(n-1)!}. \qquad (3.12)
\end{aligned}$$

Next, we will prove that (3.12) holds for $n+1$. By the expression of $F^n(\tau)$, there exists a constant L large enough, such that

$$|F^n(\tau)| \le LMK^n \frac{(2\tau)^n}{n!}. \qquad (3.13)$$

According to (3.11), we have

$$\begin{aligned}
&|G^{s,n+1}(\xi,\eta)|\\
&\le \int_0^\eta \int_0^y LMK^n \frac{(2\tau)^n}{n!} d\tau dy\\
&+ \|b_1\|_{L^\infty} \int_\eta^\xi \int_0^\eta |G^n| ds d\tau + \|b_5\|_{L^\infty} \int_\eta^\xi \int_0^\eta |G^{s,n}| ds d\tau\\
&\le LMK^n \left(\int_0^\eta \int_0^y \frac{(2\tau)^n}{n!} d\tau dy + \int_\eta^\xi \int_0^\eta \frac{(\tau+s)^{(n+1)}}{(n+1)!} ds d\tau \right)\\
&\le MK^{n+1} \frac{(\xi+\eta)^{n+2}}{(n+2)!}, \qquad (3.14)
\end{aligned}$$

where K is chosen large enough. Next we estimate

$$\begin{aligned}
&|G^{n+1}(\xi,\eta)|\\
&\le LMK^n \frac{(\xi+\eta)^{n+1}}{(n+1)!} + \int_0^\eta \int_0^y 2LMK^n \frac{(2\tau)^n}{n!} d\tau dy\\
&+ \|b_2\|_{L^\infty} \int_\eta^\xi \int_0^\eta |G^n| ds d\tau + \|b_3\|_{L^\infty} \int_\eta^\xi \int_0^\eta |G^{s,n}| ds d\tau\\
&+ \|b_1\|_{L^\infty} \int_\eta^\xi \int_0^\eta (|G^{s,n}_{\xi\xi}| + 2|G^{s,n}_{\xi\eta}| + |G^{s,n}_{\eta\eta}|) ds d\tau\\
&+ \|b_4\|_{L^\infty} \int_\eta^\xi \int_0^\eta (|G^{s,n}_\xi| + |G^{s,n}_\eta|) ds d\tau\\
&\le MK^{n+1} \frac{(\xi+\eta)^{n+1}}{(n+1)!}, \qquad (3.15)
\end{aligned}$$

where K is chosen large enough. In a very similar way, we can obtain

$$\begin{aligned}
&|G^{n+1}_\xi|, |G^{n+1}_\eta| \le MK^{n+1} \frac{(\xi+\eta)^n}{n!},\\
&|G^{s,n+1}_\xi|, \ |G^{s,n+1}_\eta| \le MK^{n+1} \frac{(\xi+\eta)^{n+1}}{(n+1)!},\\
&|G^{s,n+1}_{\xi\eta}|, \ \ |G^{s,n+1}_{\xi\xi}|, \ |G^{s,n+1}_{\eta\eta}| \le MK^{n+1} \frac{(\xi+\eta)^n}{n!}. \qquad (3.16)
\end{aligned}$$

Thus, by induction we have proved that (3.12) holds with a constant K large enough only dependent on some known functions. Once the estimates (3.12) are proved, it follows that the solutions (3.3) and (3.4) are given by the series

$$G^s(\xi,\eta) = \sum_{n=1}^{\infty} G^{s,n}(\xi,\eta), \quad G(\xi,\eta) = \sum_{n=1}^{\infty} G^n(\xi,\eta)$$

which are two continuous functions. By the fact that $b_i(i=1,2,3,4,5)$ are continuous functions, we can obtain that $G(\cdot,\cdot)$, $G^s(\cdot,\cdot)$ and $M(\cdot)$ belongs to C^2 in terms of (3.1), (3.2) and (3.7). Hence, we obtain the following existence theorem which shows the existence of the gain functions $k(x,y)$, $s(x,y)$ and $M(x)$.

Theorem 3.1 *Let* $\lambda(\cdot) \in C^2([0,1])$, $\beta(\cdot) \in C([0,1])$, $0 < c(\cdot) \in C([0,1])$ *and* $0 < d(\cdot) \in C^1([0,1])$, *then system* (2.13), (2.14) *and* (2.15) *has a unique classical solution* $M(\cdot) \in C^2([0,1])$, $k(\cdot,\cdot)$ *and* $s(\cdot,\cdot) \in C^2(\mathbb{T})$.

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