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Analysis of Predictor Feedback for Time-Varying Delays that may Assume Zero Value

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***Abstract*— For linear systems with a time-varying input delay, the predictor feedback controller and exponential stability have been established. However, the now-classical approach of representing the delay by a transport partial differential equation (PDE) on a strictly positive and constant spatial domain precludes the possibility of the delay assuming the zero value at any time instant. To eliminate this limitation, we provide a new representation of the delay by a transport equation with a time-varying spatial domain. The resulting backstepping approach leads to the same predictor feedback that was previously designed by the last author. However, the controller derivation and the stability analysis are quite different, even though both the controller and the assumptions are the same. A representative example is provided to illustrate the methodology and results.**

***Index Terms*— Time-varying delay, coupled transport PDE-ODE, backstepping, predictor, time-varying spatial domain**

I. INTRODUCTION

Time-varying delay systems are present in numerous practical applications, such as networked control systems [1, 2] and driving control systems [3]. Several existing techniques for compensating time-varying input delay, such as [4-9], are extensions of the Smith Predictor [10].

Another efficiency approach is backstepping method was given in [11, 12] and their references. Similar to the processing approach in dealing with single input systems with time-invariant delay [13], the delay is modeled by a transport partial differential equation. The delay system is represented by a coupled PDE-ordinary differential equation (ODE) system. Then, a backstepping transformation is used to convert the original system into a stable target system. The feedback control law is obtained according to the backstepping transformation and the boundary condition of the target system. The closed-loop system is proved exponential stable by constructing a Lyapunov functional. The transport PDE provided in [11] is used in other systems widely. For general nonlinear systems with time-varying input and state delays, a global asymptotic stable predictor-based feedback controller can be designed [14]. For linear systems with time-varying input delay and additive disturbances, [15] shows that the basic predictor feedback control law is inverse optimal and establishes its robustness. For linear time-varying systems with time-varying measurement delay, [16] establishes the exponential stability of the estimation error for arbitrarily large time-varying delays.

The key challenge in dealing with time-varying delay is the selection of a state for a transport PDE, which has a non-constant propagation speed [11]. In fact, the propagation speed function in [11] needs to be uniformly bounded from below and from above by finite constants, and accordingly, the delay cannot be zero for any time. The scope of application of that method reduces greatly. This paper gives a new method to deal with the systems with time-varying input delay. Note that the propagation of first-order hyperbolic PDE is unidirectional, we can model the time-varying input delay by a first-order hyperbolic PDE with unfixed boundary, which is different from the construction given in [11]. The PDE's propagation speed is one minus the derivative of the input delay and the delay can be zero in some interval or at some points. Then, the backstepping transformation is presented in detail and a feedback controller is obtained. In addition, the relationship between backstepping transformation and predictor is discussed in a theorem.

This paper is organized as follows. Section II formulates the problem, constructs a coupled system, and designs a backstepping controller. Section III provides an analysis of the stability of the closed-loop system. Section IV discusses the relationship between the predictor and a backstepping transformation. A numerical example is provided in Section V to illustrate the results. And a conclusion is drawn in Section VI.

II. THE CONTROLLER DESIGN

Consider the following linear time-invariant system (LTI)

$$\dot{X}(t) = AX(t) + BU(\varphi(t)) \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ are the corresponding system and control matrices, $U(\cdot)$ is the control input, and $\varphi(t)$ is defined as $\varphi(t) = t - d(t)$, $d(t)$ represents the time-varying delay satisfies $0 \le d(t) < D$.

The following two assumptions hold.

Assumption 1: The time delay $d(t)$ is a continuously differentiable function and satisfies

$$\sup_{t\ge 0} \dot{d}(t) < 1$$

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Remark 1. Assumption 1 indicates that the growth rate of time delay does not exceed that of time itself, and thus the invertibility of $\varphi(t)$ is guaranteed.

We define

$$\pi_0^* = \frac{1}{\inf_{t\geq 0} \dot{\varphi}(t)} \tag{2}$$

$$\pi_1^* = \sup_{t\geq 0}(\varphi^{-1}(t) - t) \tag{3}$$

According to Assumption 1, we have $\pi_0^* > 0$ and $\pi_1^* < +\infty$.

Assumption 2: The pair (A,B) is stabilizable, namely, there exists a vector $K \in \mathbb{R}^{1\times n}$ such that $A+BK$ is Hurwitz.

When $t \geq \varphi^{-1}(0)$, the delay-input $U(\varphi(t))$ can be modeled by the following first-order hyperbolic PDE

$$\begin{cases} u_t(x,t) = \dot{\varphi}(t) u_x(x,t) \\ u(d(t),t) = U(t) \end{cases} \tag{4}$$

where $(x,t) \in [0,d(t)] \times (\varphi^{-1}(0), +\infty)$. The solution of (4) is given as

$$u(x,t) = U(x+\varphi(t))$$

Consequently, (1) can be transformed into following PDE-ODE coupled system

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t) \\ u_t(x,t) = \dot{\varphi}(t) u_x(x,t) \\ u(d(t),t) = U(t) \end{cases} \tag{5}$$

If the system (5) can be stabilized by the input $U(t)$ which acts at the right end of the PDE, then the system (1) can be also stabilized by $U(t)$.

The main procedure is to find a state transformation and a state feedback control law that convert the system (5) into a stable target system. We employ the following target system:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t) \\ w_t(x,t) = \dot{\varphi}(t) w_x(x,t) \\ w(d(t),t) = 0 \end{cases} \tag{6}$$

where K is a vector such that $A+BK$ is Hurwitz.

In (6), $w(0,t) = 0$ for any $t \geq \varphi^{-1}(0)$. Hence, the state $X(t)$ satisfies

$$\dot{X}(t) = (A+BK)X(t), \quad t \geq \varphi^{-1}(0)$$

Thus, the plant obeys the nominal closed-loop system after $t = \varphi^{-1}(0)$. The stability with respect to an appropriate norm will be discussed later.

If $\dot{\varphi}(t) \equiv 1$, that is to say $d(t)$ is equal to a constant, we can use the standard backstepping transformation proposed in [17]. The main difference here is that the speed of propagation PDE as well as the spatial domain are time-varying due to the time-varying delay. We propose a new transformation.

Consider the backstepping transformation $(X,u) \to (X,w)$

$$\begin{cases} X(t) = X(t) \\ w(x,t) = u(x,t) - \Phi(x,t)X(t) - \int_0^x \Gamma(x,y,t)u(y,t)\mathrm{d}y \end{cases} \tag{7}$$

which could convert (5) into the target system (6). $\Phi(x,t)$ and $\Gamma(x,y,t)$ are kernels to be determined.

Taking the derivatives of $w(x,t)$ with respect to t and x, we have

$$\begin{aligned} w_t(x,t) = {} & \dot{\varphi}(t)u_x(x,t) - [\Phi_t(x,t) + \Phi(x,t)A]X(t) \\ & - [\Phi(x,t)B - \dot{\varphi}(t)\Gamma(x,0,t)]u(0,t) \\ & - \int_0^x [\Gamma_t(x,y,t) - \dot{\varphi}(t)\Gamma_y(x,y,t)]u(y,t)\mathrm{d}y \\ & - \dot{\varphi}(t)\Gamma(x,x,t)u(x,t) \end{aligned}$$

where the integration by parts is used, and

$$\begin{aligned} w_x(x,t) = {} & u_x(x,t) - \Phi_x(x,t)X(t) \\ & - \int_0^x \Gamma_x(x,y,t)u(y,t)\mathrm{d}y - \Gamma(x,x,t)u(x,t) \end{aligned}$$

Consider the arbitrariness of $X(t)$, $u(0,t)$, and $u(t)$, we obtain the sufficient conditions for the second equation of (6) to hold as shown in the following three formulas

$$\Phi_t(x,t) - \dot{\varphi}(t)\Phi_x(x,t) + \Phi(x,t)A = 0 \tag{8}$$

$$\Phi(x,t)B - \dot{\varphi}(t)\Gamma(x,0,t) = 0 \tag{9}$$

$$\dot{\varphi}(t)[\Gamma_x(x,y,t) + \Gamma_y(x,y,t)] - \Gamma_t(x,y,t) = 0 \tag{10}$$

To find the boundary condition for (8), let us set $x = 0$ in the second equation in (7), which gives

$$u(0,t) = w(0,t) + \Phi(0,t)X(t) \tag{11}$$

Substituting (11) into the first equation in (5), we get

$$\dot{X}(t) = (A + B\Phi(0,t))X(t) + Bw(0,t) \tag{12}$$

Comparing (12) with the first equation in (6), we have

$$\Phi(0,t) = K \tag{13}$$

By characteristic line method for solving first order linear PDE we can get the solution to (8) and (13) as

$$\Phi(x,t) = Ke^{(\varphi^{-1}(x+\varphi(t))-t)A} \tag{14}$$

Substituting (14) into (9), we have

$$\Gamma(x,0,t) = \frac{1}{\dot{\varphi}(t)} Ke^{(\varphi^{-1}(x+\varphi(t))-t)A}B \tag{15}$$

By characteristic line method for solving first order linear PDE we get the solution to (10) with the boundary condition (15) as

$$\begin{aligned} \Gamma(x,y,t) = {} & \frac{1}{\dot{\varphi}(\varphi^{-1}(y+\varphi(t)))} K \\ & \times e^{(\varphi^{-1}(x+\varphi(t)) - \varphi^{-1}(y+\varphi(t)))A} B \end{aligned}$$

We can now plug the kernels $\Phi(x,t)$ and $\Gamma(x,y,t)$ into (7) to get the backstepping transformation as

$$\begin{cases} X(t) = X(t) \\ w(x,t) = u(x,t) - Ke^{(\varphi^{-1}(x+\varphi(t))-t)A}X(t) \\ \quad - \int_0^x \frac{1}{\dot{\varphi}(\varphi^{-1}(y+\varphi(t)))} K \\ \quad \times e^{(\varphi^{-1}(x+\varphi(t)) - \varphi^{-1}(y+\varphi(t)))A} Bu(y,t)\mathrm{d}y \end{cases} \tag{16}$$

Setting $x=d(t)$ in (16), we get the control law

$$U(t)=Ke^{(\varphi^{-1}(t)-t)A}X(t)+\int_0^{d(t)}\frac{1}{\dot{\varphi}(\varphi^{-1}(y+\varphi(t)))}\times Ke^{(\varphi^{-1}(t)-\varphi^{-1}(y+\varphi(t)))A}Bu(y,t)\mathrm{d}y \quad (17)$$

III. STABILITY ANALYSIS OF THE CLOSED-LOOP SYSTEM

In the stability analysis we will use a Lyapunov construction and the backstepping transformation as well as its inverse transformation. We introduce the inverse of the backstepping transformation $(X,w)\to(X,u)$

$$\begin{cases} X(t)=X(t) \\ u(x,t)=w(x,t)+Ke^{(\varphi^{-1}(x+\varphi(t))-t)(A+BK)}X(t) \\ \quad +\int_0^x \frac{1}{\dot{\varphi}(\varphi^{-1}(y+\varphi(t)))}Ke^{(\varphi^{-1}(x+\varphi(t))-\varphi^{-1}(y+\varphi(t)))(A+BK)} \\ \quad \times Bw(y,t)\mathrm{d}y \end{cases} \quad (18)$$

The solving process of transformation (18) is to give the structural form first, and then to get the kernel function by the characteristic line method.

The stability result is given by the following theorem.

Theorem 1. Let Assumptions 1 and 2 hold, the initial condition $u(x,0)=\phi_0(-x)$, $x\in[0,d(0)]$, and $X_0=X(0)$, where $\phi_0(\cdot)$ is the initial input of $U(\cdot)$. The closed-loop system consisting of the plant (5) with the controller (17) is exponentially stable at the origin in the sense of the norm

$$\Psi(t)=|X(t)|^2+\int_0^{d(t)}u^2(x,t)\mathrm{d}x \quad (19)$$

Namely, there are positive constant G and μ such that

$$\Psi(t)\le Ge^{-\mu t}\Psi(0)$$

where

$$\Psi(0)=|X_0|^2+\int_0^{d(0)}u^2(x,0)\mathrm{d}x$$

Proof. First we prove that the origin of the target system (6) is exponentially stable. Consider a Lyapunov function

$$V(t)=X^T(t)PX(t)+\frac{a}{2}\int_0^{d(t)}e^x w^2(x,t)\mathrm{d}x \quad (20)$$

where $P=P^T>0$ is the solution to the Lyapunov equation

$$P(A+BK)+(A+BK)^TP=-Q$$

for some $Q=Q^T>0$, and the parameter a is a positive constant to be chosen later. We have

$$\dot{V}(t)=-X^T(t)QX(t)+2X^T(t)PBw(0,t)-\frac{a}{2}\dot{\varphi}(t)w^2(0,t)-\frac{a}{2}\int_0^{d(t)}e^x\dot{\varphi}(t)w^2(x,t)\mathrm{d}x$$

Since

$$2X^T(t)PBw(0,t)-\frac{a}{2}\dot{\varphi}(t)w^2(0,t)\le\frac{2}{a\dot{\varphi}(t)}\left|X^T(t)PB\right|^2$$

we have

$$\dot{V}(t)\le -X^T(t)QX(t)+\frac{2}{a\dot{\varphi}(t)}X^T(t)PBB^TPX(t)-\frac{a\dot{\varphi}(t)}{2}\int_0^{d(t)}e^x w^2(x,t)\mathrm{d}x$$

Choose $a=\frac{4\lambda_{\max}(PBB^TP)\pi_0^*}{\lambda_{\min}(Q)}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the minimum and maximum eigenvalues of the corresponding matrices, π_0^* is defined as in (2). Then

$$\begin{aligned}\dot{V}(t)&\le-\frac{\lambda_{\min}(Q)}{2}|X(t)|^2-\frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}\int_0^{d(t)}e^x w^2(x,t)\mathrm{d}x\\&\le-\mu V(t)\end{aligned}$$

where

$$\mu=\min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},\frac{1}{\pi_0^*}\right\}$$

Thus we obtain

$$V(t)\le e^{-\mu t}V(0),\quad \forall t\ge 0 \quad (21)$$

Let us now denote

$$\Omega(t)=|X(t)|^2+\int_0^{d(t)}w^2(x,t)\mathrm{d}x \quad (22)$$

Combining (20) and (22), we have that

$$\alpha_1\Omega(t)\le V(t)\le\alpha_2\Omega(t) \quad (23)$$

where

$$\alpha_1=\min\left\{\lambda_{\min}(P),\frac{a}{2}\right\}$$
$$\alpha_2=\max\left\{\lambda_{\max}(P),\frac{a}{2}e^D\right\}$$

According to (21) and (23), it follows that

$$\Omega(t)\le\frac{1}{\alpha_1}V(t)\le\frac{e^{-\mu t}}{\alpha_1}V(0)\le\frac{e^{-\mu t}\alpha_2}{\alpha_1}\Omega(0) \quad (24)$$

Now we consider the norm (19). From the backstepping transformation (16), we get

$$\begin{aligned}&\int_0^{d(t)}w^2(x,t)\mathrm{d}x\\&\le 3\int_0^{d(t)}\Big\{u^2(x,t)+\left[Ke^{(\varphi^{-1}(x+\varphi(t))-t)A}X(t)\right]^2\\&+\int_0^x\left[\frac{1}{\dot{\varphi}(\varphi^{-1}(y+\varphi(t)))}Ke^{(\varphi^{-1}(x+\varphi(t))-\varphi^{-1}(y+\varphi(t)))A}B\right]^2\\&\times u^2(y,t)\mathrm{d}y\Big\}\,\mathrm{d}x\end{aligned}$$

The function $\varphi^{-1}(\cdot)$ is strictly increasing since $\varphi(\cdot)$ is strictly increasing. We have

$$\varphi^{-1}(x+\varphi(t))\le\varphi^{-1}(\varphi(t)+d(t))=\varphi^{-1}(t)$$

for $x\in[0,d(t)]$, and

$$\varphi^{-1}(y+\varphi(t))\ge\varphi^{-1}(\varphi(t))=t$$

for $y\in[0,x]$.

Thus,

$$
\begin{aligned}
&\int_0^{d(t)} w^2(x,t)\mathrm{d}x \\
&\le 3\int_0^{d(t)} \Big\{u^2(x,t) + |K|^2 e^{2(\varphi^{-1}(t)-t)|A|}|X(t)|^2 \\
&+ \int_0^x (\pi_0^*)^2 |K|^2 e^{2(\varphi^{-1}(t)-t)|A|}|B|^2 u^2(y,t)\mathrm{d}y\Big\}\mathrm{d}x \\
&\le 3\int_0^{d(t)} \Big\{u^2(x,t) + |K|^2 e^{2\pi_1^*|A|}|X(t)|^2 \\
&+ \int_0^{d(t)} (\pi_0^*)^2 |K|^2 e^{2\pi_1^*|A|}|B|^2 u^2(y,t)\mathrm{d}y\Big\}\mathrm{d}x \\
&\le 3\left(1 + D(\pi_0^*)^2|K|^2 e^{2\pi_1^*|A|}|B|^2\right)\int_0^{d(t)} u^2(x,t)\mathrm{d}x \\
&+ 3D|K|^2 e^{2\pi_1^*|A|}|X(t)|^2 \\
&= \beta_1\int_0^{d(t)} u^2(x,t)\mathrm{d}x + \beta_2|X(t)|^2
\end{aligned}
$$

where π_1^* is defined as (3), and

$$
\begin{aligned}
\beta_1 &= 3\left(1 + D(\pi_0^*)^2|K|^2 e^{2\pi_1^*|A|}|B|^2\right) \\
\beta_2 &= 3D|K|^2 e^{2\pi_1^*|A|}
\end{aligned}
$$

Similarly, from the inverse backstepping transformation (18), we get

$$
\begin{aligned}
&\int_0^{d(t)} u^2(x,t)\mathrm{d}x \\
&\le \gamma_1\int_0^{d(t)} w^2(x,t)\mathrm{d}x + \gamma_2|X(t)|^2
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1 &= 3\left(1 + D(\pi_0^*)^2|K|^2 e^{2\pi_1^*|A+BK|}|B|^2\right) \\
\gamma_2 &= 3D|K|^2 e^{2\pi_1^*|A+BK|}
\end{aligned}
$$

With a few substitutions we obtain that

$$
\begin{aligned}
\Omega(t) &= |X(t)|^2 + \int_0^{d(t)} w^2(x,t)\mathrm{d}x \\
&\le \beta_1\int_0^{d(t)} u^2(x,t)\mathrm{d}x + (1+\beta_2)|X(t)|^2 \\
&\le \max\{\beta_1, 1+\beta_2\}\Psi(t) \\
\Psi(t) &= |X(t)|^2 + \int_0^{d(t)} u^2(x,t)\mathrm{d}x \\
&\le \gamma_1\int_0^{d(t)} w^2(x,t)\mathrm{d}x + (1+\gamma_2)|X(t)|^2 \\
&\le \max\{\gamma_1, 1+\gamma_2\}\Omega(t)
\end{aligned}
$$

Namely,

$$
\sigma_1\Psi(t) \le \Omega(t) \le \sigma_2\Psi(t) \tag{25}
$$

where

$$
\begin{aligned}
\sigma_1 &= \frac{1}{\max\{\gamma_1, 1+\gamma_2\}} \\
\sigma_2 &= \max\{\beta_1, 1+\beta_2\}
\end{aligned}
$$

Finally, with (24) and (25) we get

$$
\Psi(t) \le \frac{1}{\sigma_1}\Omega(t) \le \frac{e^{-\mu t}\alpha_2}{\alpha_1\sigma_1}\Omega(0) \le \frac{e^{-\mu t}\alpha_2\sigma_2}{\alpha_1\sigma_1}\Psi(0), \quad \forall t \ge 0
$$

Let $G = \frac{\alpha_2\sigma_2}{\alpha_1\sigma_1}$, we complete the proof of the theorem.

IV. THE RELATIONSHIP BETWEEN PREDICTOR AND BACKSTEPPING TRANSFORMATION

The controller (17) is given in terms of the transport PDE state $u(x,t)$. Recall that

$$
u(x,t) = U(x+\varphi(t))
$$

where

$$
(x,t) \in [0, d(t)] \times (\varphi^{-1}(0), +\infty)
$$

Replacing $u(y,t)$ by $U(y+\varphi(t))$ in (17) and setting $\xi = y+\varphi(t)$, we have

$$
\begin{aligned}
U(t) &= Ke^{(\varphi^{-1}(t)-t)A}X(t) \\
&+ \int_{\varphi(t)}^t \frac{1}{\dot{\varphi}(\varphi^{-1}(\xi))} Ke^{(\varphi^{-1}(t)-\varphi^{-1}(\xi))A}BU(\xi)\mathrm{d}\xi
\end{aligned} \tag{26}
$$

Consider the system (1) again. The main premise of the predictor based design is that one generates the controller

$$
U(\varphi(t)) = KX(t), \quad \forall t \ge \varphi^{-1}(0) \tag{27}
$$

so that the closed-loop system is

$$
\dot{X}(t) = (A+BK)X(t), \quad \forall t \ge \varphi^{-1}(0)
$$

The gain vector K is selected so that $A+BK$ is Hurwitz.

We now rewrite (27) as

$$
U(t) = KX(\varphi^{-1}(t)), \quad \forall t \ge 0 \tag{28}
$$

With the variation of constants formula to the model (1), the quality $X(\varphi^{-1}(t))$ for all $t \ge 0$ is written as

$$
\begin{aligned}
X(\varphi^{-1}(t)) &= e^{(\varphi^{-1}(t)-t)A}X(t) \\
&+ \int_{\varphi(t)}^t \frac{1}{\dot{\varphi}(\varphi^{-1}(\xi))} e^{(\varphi^{-1}(t)-\varphi^{-1}(\xi))A}BU(\xi)\mathrm{d}\xi
\end{aligned}
$$

Substituting this expression into (28), we obtain the predictor controller (26). That is the controller $U(t)$ uses the $\varphi^{-1}(t)-t$ time units ahead predictor of X.

Denote

$$P(\theta)=X(\varphi^{-1}(\theta)), \quad \varphi(t)\le\theta\le t, \quad \forall t\ge 0 \tag{29}$$

We have the following theorem.

Theorem 2. The predictor state $P(\theta)$, for all $\theta \ge \varphi(0)$ can be described equivalently as

$$p(x,t)=e^{(\varphi^{-1}(x+\varphi(t))-t)A}X(t) + \int_0^x \frac{1}{\dot\varphi(\varphi^{-1}(y+\varphi(t)))} e^{(\varphi^{-1}(x+\varphi(t))-\varphi^{-1}(y+\varphi(t)))A}Bu(y,t)\mathrm{d}y, \quad (x,t)\in[0,d(t)]\times\mathbb{R}^+ \tag{30}$$

Furthermore, the backstepping transformation (16) can be rewritten as

$$\begin{cases} X(t)=X(t) \\ w(x,t)=u(x,t)-Kp(x,t) \end{cases}, (x,t)\in[0,d(t)]\times\mathbb{R}^+ \tag{31}$$

Proof: The function p satisfies the following ODE in x:

$$p_x(x,t)=\frac{1}{\dot\varphi(\varphi^{-1}(x+\varphi(t)))}(Ap(x,t)+Bu(x,t)) \tag{32}$$

With initial condition

$$p(0,t)=X(t) \tag{33}$$

The solution to (32) and (33) is

$$p(x,t)=X(\varphi^{-1}(x+\varphi(t))), \quad (x,t)\in[0,d(t)]\times\mathbb{R}^+ \tag{34}$$

In order to show this, first note that (34) satisfies the boundary condition (33). Then, the function $X(\varphi^{-1}(x+\varphi(t)))$ also satisfies the ODE (32) in x which follows from the fact that by (1) one can conclude that for $(x,t)\in[0,d(t)]\times\mathbb{R}^+$:

$$\begin{aligned} &\dot X(\varphi^{-1}(x+\varphi(t))) \\ &=\frac{1}{\dot\varphi(\varphi^{-1}(x+\varphi(t)))}\big(AX(\varphi^{-1}(x+\varphi(t))) \\ &+BU(x+\varphi(t))\big) \end{aligned}$$

The result follows from the uniqueness of solution to the ODE (1), where $u(x,t)=U(x+\varphi(t))$ is used. Combining the fact $P(t)=X(\varphi^{-1}(t))$ in (29), as well as $P(x+\varphi(t))=X(\varphi^{-1}(x+\varphi(t)))$, one can conclude that:

$$P(x+\varphi(t))=p(x,t), \quad (x,t)\in[0,d(t)]\times\mathbb{R}^+ \tag{35}$$

Performing the change of variables $x=\theta-\varphi(t)$ for all $\varphi(t)\le\theta\le t$ in (30) and using (33), (34), and (35) we arrive at

$$\begin{aligned} P(\theta)&=e^{(\varphi^{-1}(\theta)-t)A}X(t) \\ &+\int_{\varphi(t)}^{\theta}\frac{1}{\dot\varphi(\varphi^{-1}(\xi))}e^{(\varphi^{-1}(\theta)-\varphi^{-1}(\xi))A}BU(\xi)\mathrm{d}\xi, \\ &\varphi(t)\le\theta\le t \end{aligned}$$

On the other hand, combining (16) and (30) we get (31). The proof is completed.

Remark 2. Theorem 2 shows that, in the backstepping transformation, there is a constant-times of the predictor of the ODE state $X(t)$ difference between the PDE state $u(x,t)$ in the original system and the PDE state $w(x,t)$ in the target system. Further more, the constant is gain vector K.

V. NUMERICAL SIMULATION

In this section, an example is prepared to verify the theory.

Example 1. Consider a one-order example described as (1), where $A=B=1$, $d(t)=(1+sint)/2$, $d(t)$ is shown in Fig. 1. Accordingly, $\varphi(t)=t-(1+sint)/2$ and φ is invertible. Taking $K=-2$, we get the simulation result shown in Fig. 3.

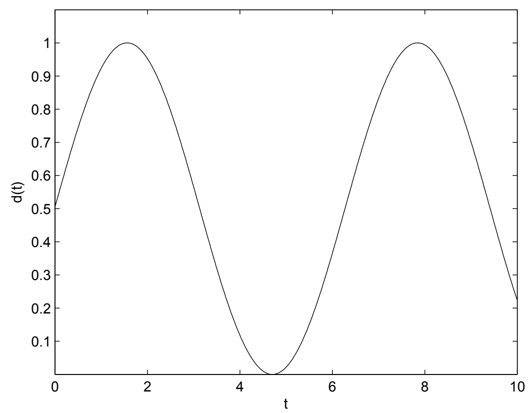

Fig. 1. The input delay

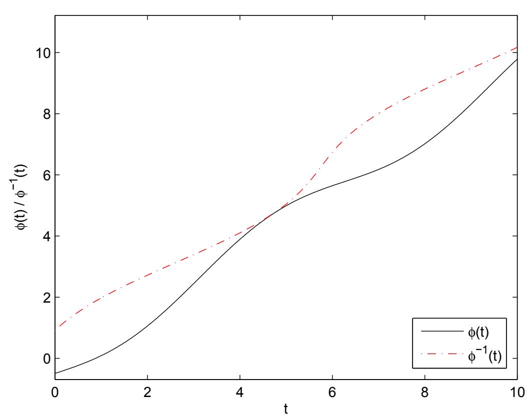

Fig. 2. $\varphi(t)$ and its inverse $\varphi^{-1}(t)$

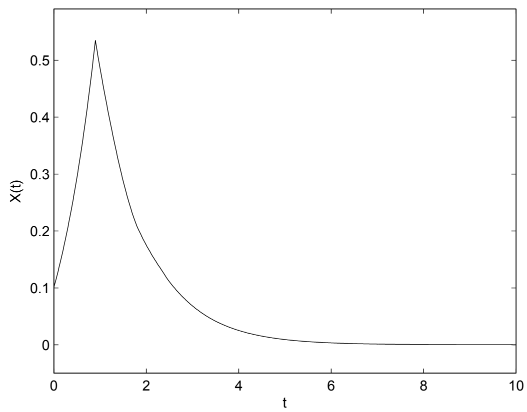

Fig. 3. State signal evolution for Example 1.

Fig. 1 shows that the graph of the delay get to zero between 4 and 5. In this case, the controller (37) still applicable. In fact, when $d(t) = 0$ the controller get be

$$U(t) = Ke^{(\varphi^{-1}(t)-t)A}X(t) \tag{36}$$

VI. CONCLUSION

This paper provides a new method of designing backstepping controller for ODE with a time-varying input delay. The derivation employs a transport equation with time-varying boundary. As a result, the time delay could be zero in some intervals or points, and the derivations of the backstepping transformation and its inverse as well as the stability analysis are given by $\varphi(t)$ rather than using a new variable "propagation speed function $\pi(x,t)$" in [11]. As a result, the application range of the controller is extended and this method could be applied to multiple time-varying delay systems that we will discuss in future work.

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