Received: 4 March 2021 | Revised: 31 January 2022 | Accepted: 1 February 2022

DOI: 10.1002/rnc.6145

RESEARCH ARTICLE

WILEY

Boundary control of a coupled Burgers' PDE-ODE system

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Abstract

This article presents a result of stabilization of a coupled partial differential equation (PDE) and ordinary differential equation (ODE) system through boundary control. The PDE is the Burgers' equation, which is a widely considered nonlinear PDE, partially due to its low order and partially due to its structure analogous to the Navier–Stokes equation, which describes fluid dynamics. The controller we employ for stabilizing this system was first developed from the boundary control problem of the corresponding linearized system, based on an infinite-dimensional backstepping transformation. The stabilization result is achieved using only one boundary measurement and one boundary control. Numerical simulations show the boundary control law can be used to stabilize the system.

KEYWORDS

boundary control, distributed parameter systems, Lyapunov methods, nonlinear systems, PDE-ODE systems

1 | INTRODUCTION

Engineering problems, due to the complex nature, are in general described by coupled systems. In recent year, coupled PDE-ODE systems have been an active area of research. Extensive examples can be found, to name just a few, in model of flexible cable in an overhead crane,[1] automated managed pressure drilling from floating vessels,[2,3] cancer cell invasion,[4] and thermal-electrochemical model in battery management systems.[5,6] This article addresses control and estimation of a coupled Burgers' PDE-ODE system. The nonlinearity in the Burgers' equation added complexity in the estimation and control design.

Burgers' equation is usually considered as a one-dimensional Navier–Stokes equation. The equation contains a nonlinear term and it is the simplest equation for which the solutions can develop shock waves. Using the Hopf-Cole transformation,[7] one can change the equation into a linear parabolic equation. Therefore, the solution do not exhibit chaotic features like sensitivity with respect to initial condition. The Burgers' equation is used in many areas of mathematical physics such as nonlinear acoustic, gas dynamics, and traffic flow. It takes the following form

$$u_t(x,t) = \epsilon u_{xx}(x,t) - u(x,t)u_x(x,t), \tag{1}$$

where $\epsilon > 0$ is called the viscosity coefficient, due to its application in fluid dynamics. Burgers' equation is an example of a semilinear PDE combining both nonlinear propagation and diffusive effects. Early research on control of Burgers' equation can be found in References 8–13, where its numerical simulation are presented in References 9,14,15. In the former references, the authors derived the nonlinear boundary control laws that achieve global asymptotic stability using the Lyapunov method. Furthermore, for some of the control laws that would require measurements in the interior of the domain, an observer-based control was developed. In Reference 16, the unstable shock-like equilibrium profile of the viscous Burgers' equation was stabilized using control at the boundaries. The explicit nonlinear full-state control law achieved exponential stability. Recent results on control of Burgers' equation can be found in Reference 17–19. Some real applications of Burgers' equation in fluid-particle system can be found in References 20 and 21, while its well-posedness problem has been discussed in Reference 22 and 23.

Stabilization of PDE systems with boundary control was considered as a challenging topic until the past two decades. With the introduction of the infinite-dimensional backstepping, a systematic method for control design and estimation of PDEs, it becomes an emerging research area.[24] The infinite-dimensional backstepping method has been successfully used for control and estimation for many PDEs, such as the Korteweg-de Vries equation,[25] Benjamin–Bona–Mahony equation,[26] Schrodinger equation,[27] Ginzburg-Landau equation,[28] parabolic PDEs,[29] and 2×2 linear hyperbolic PDEs.[30,31] In engineering problems, the backstepping method has found several applications, such as to find an optimal oil rate under gas coning condition,[32] flow control in porous media,[33] stabilization of slugging in drilling,[34] and lost circulation and kick control.[35] While most of the early efforts focus on linear PDE systems and coupled systems of a linear PDE and a linear ODE, for example,[36,37] there are only a few working on the systems with nonlinearity. Indeed, until now the infinite-dimensional backstepping is limited to Volterra nonlinearities.[38,39] Despite of this limitation, local stabilization of nonlinear PDE systems has shown promising results in recent years. For example, feedback control design of nonlinear PDEs was presented in Reference 40, which uses the backstepping technique and achieved local stabilization for a coupled system of two heterodirectional hyperbolic PDEs, called a 2×2 quasilinear hyperbolic system.

In Reference 41, a stabilizing boundary controller and an observer for a coupled heat PDE—linear ODE have been designed using the backstepping method. Since the heat equation can be considered as a linearized Burgers' equation, in this article we employ the controller for the coupled heat PDE—linear ODE. In Reference 42, a linear coupled hyperbolic PDE-ODE system has also been studied using the bakstepping method. Here, the ODE state was considered as a disturbance source for the hyperbolic PDE. It was shown that the control law is able to attenuate the disturbance. Together with the observer, they solve the output-feedback regulation problems. Other works in control of coupled PDE-ODE systems include control with Neumann interconnections,[43] delay systems,[44–46] coupled ODE-Schrodinger equation,[47] and adaptive control of PDE-ODE cascade systems with uncertain harmonic disturbances.[48] Furthermore, a coupled system of nonlinear ODE—linear PDE has been also studied.[49,50] However, to the best knowledge of the authors, the only existing result for stabilizing coupled systems of a linear ODE and a nonlinear PDE was presented in Reference 51, which studied the observer design for a coupled system consisting of a linear ODE and a semilinear hyperbolic PDE.

This article presents stabilization problem for a coupled system of a (viscous) Burgers' PDE and an ODE. A preliminary version of the boundary stabilization part was presented in Reference 52. Based on the linear feedback Volterra transformation and the related controller which exponentially stabilizes the linearized Burger's PDE-ODE system in spatial $\mathbb{H}^1$ norm, we employ a strict Lyapunov functional and show the controller locally stabilizes the nonlinear Burgers' PDE-ODE system in the sense of the $\mathbb{H}^2$ norm with exponential decay rate. An observer is designed for the nonlinear system using only a boundary measurement. The boundary observer together with the state feedback controller are used in the output feedback regulation.

The article begins with a problem statement in Section 2, which is followed by a presentation of some preliminary control results for the corresponding linearized system in Section 3. A state feedback control for the coupled Burgers' PDE-ODE system is presented in Section 4, while an observer is presented in Section 5. The state feedback and state observer are used in the output feedback regulation and is presented in Section 6. To demonstrate the design, numerical simulations are presented in Section 7. Finally, we present the conclusions in Section 8.

We first establish some definitions and notations. For a vector $Z = (z_i)_{i=\overline{1,n}} \in R^n$, denote its 1-norm as $|Z| = |z_1| + \cdots + |z_n|$. For a real-valued function $f(x,t)$, where $x \in [0,1]$ and $t \in [0,\infty)$, we define the following norms

$$\|f\|_\infty = \sup_{x\in[0,1]} |f|, \tag{2}$$

$$\|f\|_{\mathbb{L}^1} = \int_0^1 |f| \, dx, \tag{3}$$

$$\|f\|_{\mathbb{L}^2} = \left(\int_0^1 f^2 \, dx \right)^{\frac{1}{2}}, \tag{4}$$

$$\|f\|_{\mathbb{H}^i} = \sum_{k=0}^{i} \left(\int_0^1 \left(\frac{\partial^i f}{\partial x^i} \right)^2 dx \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n. \tag{5}$$

For $f \in \mathbb{H}^2([0,1])$, the following inequalities hold[40]

$$\|f\|_{\mathbb{L}^1} \leq a_1 \|f\|_{\mathbb{L}^2} \leq a_2 \|f\|_\infty, \tag{6}$$

$$\|f\|_\infty \leq a_3 (\|f\|_{\mathbb{L}^2} + \|f_x\|_{\mathbb{L}^2}) \leq a_4 \|f\|_{\mathbb{H}^1}, \tag{7}$$

$$\|f_x\|_\infty \leq a_5 (\|f_x\|_{\mathbb{L}^2} + \|f_{xx}\|_{\mathbb{L}^2}) \leq a_6 \|f\|_{\mathbb{H}^2}, \tag{8}$$

where $a_i, \ i = \overline{1, 6}$, are positive constants.

2 | PROBLEM STATEMENT

Consider the following boundary control problem of a coupled Burgers' PDE-ODE system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9}$$

$$u_t(x,t) = \epsilon u_{xx}(x,t) - u(x,t)u_x(x,t) + CX(t), \tag{10}$$

$$u_x(0,t) = 0, \tag{11}$$

$$u(1,t) = U(t), \tag{12}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, and the pair (A, B), with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n$, is assumed to be stabilizable; $u(x,t) \in \mathbb{R}$ is the PDE state, and $C^\intercal \in \mathbb{R}^n$ is a constant vector; $U(t)$ is the scalar control input to the entire system. The viscosity coefficient $\epsilon \geq 0$, due to the general application of this equation to the fluid dynamics, is typically referred to as viscosity. When $\epsilon = 0$, the Burgers' equation becomes the inviscid Burgers' equation. It is an example of nonlinear conservative equation, which has a solution in the form of shock waves. Control of the inviscid Burgers' equation is a challenging problem and will not be discussed here. In this article, we consider the viscous case, that is, $\epsilon > 0$, when the open-loop PDE presents a dissipative characteristic. Without loss of generality, we set $\epsilon = 1$. System (9)–(12) can be used to model electro-hydrodynamic model in plasma physics, where $CX(t)$ represents the forcing term.[53] If C depends on the spatial variable x, then a completely different approach needs to be taken since (17) may not exist for such systems.

In the coupled system, the ODE state has a uniform influence on the PDE, as seen from (9); while on the other hand, the PDE boundary state $u(0,t)$ can also be considered as a force acting on the ODE. The block diagram in Figure 1 shows clearly the control structure, especially the bidirectional influences between the PDE and the ODE.

The objective of this article is to develop the state control input $U(t)$ to stabilize the entire coupled Burgers' PDE-ODE system (9)–(12). Furthermore, assuming we able to measure $u(0,t)$, we develop an anticollocated Luenberger observer to estimate the PDE and ODE state. The state estimations are used in the control input as the output feedback control. Note that further investigation regarding the existence of shock-like unstable equilibrium solutions for (9)–(12) need to be conducted before designing a state feedback controller for such problems. If such a solution exists, one may consider using the same approach as in Reference 16. However, one problem that may arise is that the composite transformation can lead to an unsolvable target system.

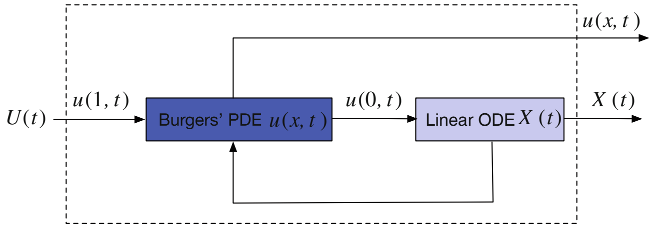


FIGURE 1 Control of a coupled Burgers' PDE-ODE system

3 | CONTROL DESIGN FOR THE COUPLED LINEARIZED BURGERS' PDE-ODE SYSTEM

The corresponding linearized system of (9)–(12) around the zero equilibrium $(u, X)^\intercal = (0, 0)^\intercal$ is given by

$$\dot{X}(t) = AX(t) + Bu(0, t), \tag{13}$$

$$u_t(x, t) = u_{xx}(x, t) + CX(t), \tag{14}$$

$$u_x(0, t) = 0, \tag{15}$$

$$u(1, t) = U(t). \tag{16}$$

Boundary stabilization by output feedback for (13)–(16) has been presented in Reference 41 and, since some of the results are used in Section 4, is highlighted in this section. The nonlinear convection term uu_x in (10) causes the shock-like unstable profile, which for some initial conditions cannot be stabilized by the boundary controller presented in Reference 41, that is, the controller for the linear system is only achieved local stabilization.

3.1 | State feedback controller design

Following Reference 41, we use the transformation

$$w(x, t) = u(x, t) - \int_0^x q(x, y)u(y, t)\, dy - \gamma(x)X(t), \tag{17}$$

to map (13)–(16) into the following system

$$\dot{X}(t) = (A + BK)X(t) + Bw(0, t), \tag{18}$$

$$w_t(x, t) = w_{xx}(x, t), \tag{19}$$

$$w_x(0, t) = 0, \tag{20}$$

$$w(1, t) = 0, \tag{21}$$

where K is chosen such that $A + BK$ is Hurwitz. The transformation kernels in (17), that is, $q(x, y)$ and $\gamma(x)$, satisfy the following system

$$q_{xx}(x, y) = q_{yy}(x, y), \tag{22}$$

$$q(x, x) = 0, \tag{23}$$

$$q_y(x,0) = -\gamma(x)B, \tag{24}$$

$$\gamma''(x) = \gamma(x)A + C\int_0^x q(x,y)\,\mathrm{d}y - C, \tag{25}$$

$$\gamma(0) = K, \tag{26}$$

$$\gamma'(0) = 0, \tag{27}$$

where $(x,y) \in \mathcal{T} = \{(x,y) : 0 \le y \le x \le 1\}$. Solving (22)–(24), the solution for the kernel $q(x,y)$ is given by

$$q(x,y) = \int_0^{x-y} \gamma(\xi)B\,\mathrm{d}\xi. \tag{28}$$

Substituting (28) into (25), we have

$$\gamma''(x) = \gamma(x)A + C\int_0^x\int_0^{x-y} \gamma(\xi)B\,\mathrm{d}\xi\,\mathrm{d}y - C. \tag{29}$$

The solution for this equation with boundary conditions (26)–(27) is given by

$$\gamma(x) = \Lambda e^{Dx}E, \tag{30}$$

where

$$\Lambda = \begin{pmatrix} K & 0 & KA - C & 0 \end{pmatrix}, \tag{31}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & BC \\ I & 0 & 0 & 0 \\ 0 & I & 0 & A \\ 0 & 0 & I & 0 \end{pmatrix}, \tag{32}$$

$$E = \begin{pmatrix} I & 0 & 0 & 0 \end{pmatrix}^{\intercal}. \tag{33}$$

Thus, we obtain

$$q(x,y) = \int_0^{x-y} \Lambda e^{D\xi}EB\,\mathrm{d}\xi. \tag{34}$$

The inverse of the transformation (17) is given by

$$u(x,t) = w(x,t) + \int_0^x p(x,y)w(y,t)\,\mathrm{d}y + \kappa(x)X(t), \tag{35}$$

where the inverse transformation kernels $p(x,y)$ and $\kappa(x)$ satisfy

$$p_{xx}(x,y) = p_{yy}(x,y), \tag{36}$$

$$p(x,x) = 0, \tag{37}$$

$$p_y(x,0) = -\kappa(x)B, \tag{38}$$

$$\kappa''(x) = \kappa(x)(A+BK) - C, \tag{39}$$

$$\kappa(0) = K, \tag{40}$$

$$\kappa'(0) = 0, \tag{41}$$

where $(x,y) \in \mathscr{T}$. The solution $\kappa(x)$ is given by

$$\kappa(x) = \left(K - C(A+BK)^{-1}\right) G(x) + C(A+BK)^{-1}, \tag{42}$$

where

$$G(x) = \begin{pmatrix} I & 0 \end{pmatrix} e^{Hx} \begin{pmatrix} I \\ 0 \end{pmatrix}, \tag{43}$$

$$H = \begin{pmatrix} 0 & (A+BK) \\ I & 0 \end{pmatrix}. \tag{44}$$

Thus, $p(x,y)$ is given by

$$p(x,y) = \int_0^{x-y} \left(\left(K - C(A+BK)^{-1}\right) G(\xi) + C(A+BK)^{-1}\right) B \, \mathrm{d}\xi. \tag{45}$$

From (16), (17), and (21), the control law U is given by

$$U(t) = \int_0^1 q(1,y)u(y,t)\, \mathrm{d}y + \gamma(1)X(t). \tag{46}$$

Substituting (30) and (34) into (46), we have

$$U(t) = \int_0^1 \left(\int_0^{1-y} \Lambda e^{D\xi} \, \mathrm{d}\xi \right) EBu(y,t)\, \mathrm{d}y + \Lambda e^{D} EX(t). \tag{47}$$

The coupled X-subsystem (18) and the w-subsystem (19)–(21) can be easily proven to be exponentially stable by using the following Lyapunov functional

$$V_1(t) = X(t)^{\intercal} PX(t) + \frac{a}{2}\|w(\cdot,t)\|^2_{\mathbb{L}^2} + \frac{1}{2}\|w(\cdot,t)\|^2_{\mathbb{H}^1}, \tag{48}$$

where $P = P^{\intercal} > 0$ is the unique solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^{\intercal} P = -Q, \tag{49}$$

for some positive definite matrix Q, and the parameter a is chosen such that $a > \frac{8|PB|^2}{\lambda_{\min}(Q)} + 2$.

Lemma 1. *Consider the system (13)–(16) with initial data* $X(0) \in \mathbb{R}^n$ *and* $u_0(x) \in \mathbb{H}^1([0,1])$ *compatible with the control law (47). The system has a unique classical solution and is exponentially stabilized in the sense of the norm*

$$\|(X(t), u(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|^2_{\mathbb{H}^1}. \tag{50}$$

3.2 Observer design

Let us assume that only $u(0,t)$ is available for measurement. An anticollocated observer for (13)–(16) is designed as follows

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(0,t) + P_0\tilde{u}(0,t), \tag{51}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + C\hat{X}(t) + p_1(x)\tilde{u}(0,t), \tag{52}$$

$$\hat{u}_x(0,t) = 0, \tag{53}$$

$$\hat{u}(1,t) = U(t), \tag{54}$$

where $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$. The observer gains P_0 and p_1 are to be determined later. Defining the ODE state error $\tilde{X}(t) = X(t) - \hat{X}(t)$, we have

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}(0,t), \tag{55}$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + C\tilde{X}(t) - p_1(x)\tilde{u}(0,t), \tag{56}$$

$$\tilde{u}_x(0,t) = 0, \tag{57}$$

$$\tilde{u}(1,t) = 0. \tag{58}$$

We use transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \Theta(x)\tilde{X}(t), \tag{59}$$

to transform (55)–(58) into the following target system

$$\dot{\tilde{X}}(t) = (A - P_0\Theta(0))\,\tilde{X}(t) - P_0\tilde{w}(0,t), \tag{60}$$

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t), \tag{61}$$

$$\tilde{w}_x(0,t) = 0, \tag{62}$$

$$\tilde{w}(1,t) = 0. \tag{63}$$

If we assume the pair $(A, \Theta(0))$ is stabilizable, then the target system (60)–(63) is exponentially stable if the gain vector P_0 is chosen such that the matrix $A - P_0\Theta(0)$ is Hurwitz and the gain function $p_1(x) = \Theta(x)P_0$. Furthermore, the row vector $\Theta(x)$ needs to satisfy

$$\Theta''(x) = \Theta(x)A - C, \tag{64}$$

$$\Theta'(0) = 0, \tag{65}$$

$$\Theta(1) = 0. \tag{66}$$

According to Reference 41, the solution to the above two-point-boundary-value problem exists and is unique if A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k \in \mathbb{N}$, and the unique solution is given by

$$\Theta(x) = \Gamma(x)e^{Jx}\begin{pmatrix} I \\ 0 \end{pmatrix}, \tag{67}$$

where

$$J = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}, \tag{68}$$

$$\Gamma(x) = \begin{pmatrix} \Theta(0) & 0 \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & C \end{pmatrix} e^{-J\xi} \, \mathrm{d}\xi, \tag{69}$$

$$\Theta(0) = \int_0^1 \begin{pmatrix} 0 & C \end{pmatrix} e^{-J\xi} \, \mathrm{d}\xi \cdot e^J I_o \left(I_o^{\mathrm{T}} e^J I_o \right)^{-1}, \tag{70}$$

$$I_o = \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{71}$$

Lemma 2. *Assume that A has no eigenvalues of the form $-(2k+1)^2\pi^2/4$ for $k \in \mathbb{N}$. If the pair $(A, \Theta(0))$ is stabilizable and the gain function $p_1(x) = \Theta(x)P_0$, where P_0 is chosen such that $A - P_0\Theta(0)$ is Hurwitz, then the observer error (55)–(58) exponentially converges to zero in the sense of the norm*

$$\left\| \left(\tilde{X}(t), \tilde{u}(\cdot, t) \right) \right\|^2 = |\tilde{X}(t)|^2 + \|\tilde{u}(\cdot, t)\|_{\mathbb{H}^1}^2. \tag{72}$$

3.3 | Output feedback controller design

We employ the following continuous and invertible transformation

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x q(x,y)\hat{u}(y,t) \, \mathrm{d}y - \gamma(x)\hat{X}(t), \tag{73}$$

then (51)–(54) is transformed into the following system

$$\dot{\hat{X}}(t) = (A + BK)\hat{X}(t) + B\hat{w}(0,t) + (B + P_0) \left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t) \right), \tag{74}$$

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \overline{p}_1(x) \left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t) \right), \tag{75}$$

$$\hat{w}_x(0,t) = 0, \tag{76}$$

$$\hat{w}(1,t) = 0, \tag{77}$$

where

$$\overline{p}_1(x) = p_1(x) - \gamma(x)(B + P_0) - \int_0^x q(x,y)p_1(y) \, \mathrm{d}y. \tag{78}$$

Replacing the state in (47) with its estimate $(\hat{u}, \hat{X})$ obtained from the designed observer (51)–(54), the output feedback control law is given by

$$U(t) = \int_0^1 \left(\int_0^{1-y} \Lambda e^{D\xi} \, \mathrm{d}\xi \right) EB\hat{u}(y,t) \, \mathrm{d}y + \Lambda e^D E\hat{X}(t). \tag{79}$$

The detailed proof for stabilization using output feedback is omitted here and can be referred to Reference 41. The idea is to use the following Lyapunov functional

$$V_2(t) = \hat{X}(t)^\intercal \hat{P}\hat{X}(t) + \frac{\hat{a}}{2}\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x + \bar{d}\left(\tilde{X}(t)^\intercal \tilde{P}\tilde{X}(t) + \frac{\tilde{a}}{2}\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x\right), \tag{80}$$

where the matrices $\hat{P} = \hat{P}^\intercal > 0$ and $\tilde{P} = \tilde{P}^\intercal > 0$ are the solutions to the Lyapunov functionals

$$\hat{P}(A + BK) + (A + BK)^\intercal\,\hat{P} = -\hat{Q}, \tag{81}$$

$$\tilde{P}(A - P_0\Theta(0)) + (A - P_0\Theta(0))^\intercal\,\tilde{P} = -\tilde{Q}, \tag{82}$$

for some $\hat{Q} = \hat{Q}^\intercal > 0$ and $\tilde{Q} = \tilde{Q}^\intercal > 0$, respectively.

Lemma 3. *Assume that A has no eigenvalues of the form $-(2k+1)^2\pi^2/4$ for $k \in \mathbb{N}$. For any initial data $X(0), \hat{X}(0) \in \mathbb{R}^n$ and $u_0(x), \hat{u}_0(x) \in \mathbb{H}^1([0,1])$ compatible with the control law (79), the closed-loop system consisting of the plant (13)–(16), the controller (79), and the observer (51)–(54) has a unique classical solution and is exponentially stabilized in the sense of the norm*

$$\left\|\left(X, u, \hat{X}, \hat{u}\right)\right\|^2 = |X|^2 + \|u\|_{\mathbb{H}^1}^2 + |\hat{X}|^2 + \|\hat{u}\|_{\mathbb{H}^1}^2. \tag{83}$$

4 | STATE FEEDBACK CONTROL OF THE COUPLED BURGERS' PDE-ODE SYSTEM

The solution of (9) is given by

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(0,\tau)\,\mathrm{d}\tau. \tag{84}$$

The idea is to apply the control law (47) to stabilize the coupled Burgers' PDE-ODE system. Substituting (84) into (10) and substituting (47) into (12), respectively, the closed-loop Burgers' PDE-ODE system is given by

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + C\left(e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(0,\tau)\,\mathrm{d}\tau\right), \tag{85}$$

$$u_x(0,t) = 0, \tag{86}$$

$$u(1,t) = \int_0^1 \left(\int_0^{1-y} \Lambda e^{D\xi}\,\mathrm{d}\xi\right) EBu(y,t)\,\mathrm{d}y + \Lambda e^{D}EX(t), \tag{87}$$

for which the following theorem holds.

Theorem 1. *Consider the closed-loop system (85)–(87) with initial data $X(0) \in \mathbb{R}^n$ and $u_0(x) \in \mathbb{H}^2([0,1])$ compatible with the control law (47). There exists $\delta > 0$ such that, if $\|u_0\|_{\mathbb{H}^2} + |X(0)| \leq \delta$, the closed-loop system has a unique classical solution and is exponentially stabilized in the sense of the norm*

$$\|(X(t), u(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|_{\mathbb{H}^2}^2. \tag{88}$$

In order to prove Theorem 1, we first define the following functionals

$$\mathcal{K}[f] = f(x,t) - \int_0^x q(x,y)f(y,t)\,\mathrm{d}y, \tag{89}$$

$$\mathcal{L}[f] = f(x,t) + \int_0^x p(x,y)f(y,t)\,\mathrm{d}y, \tag{90}$$

$$\mathcal{K}_1[f] = -q(x,x)f(x,t) + \int_0^x q_y(x,y)f(y,t)\,\mathrm{d}y, \tag{91}$$

$$\mathcal{L}_1[f] = p(x,x)f(x,t) + \int_0^x p_x(x,y)f(y,t)\,\mathrm{d}y. \tag{92}$$

Since the kernels in both the direct and inverse transformations are $C^2(\mathcal{T})$ functionals, they satisfy the following inequalities for any $(x,t) \in \mathcal{T}$

$$|\mathcal{K}[f]| \le b_1 \left(|f| + \|f\|_{\mathbb{L}^1}\right), \tag{93}$$

$$|\mathcal{L}[f]| \le b_2 \left(|f| + \|f\|_{\mathbb{L}^1}\right), \tag{94}$$

$$|\mathcal{K}_1[f]| \le b_4 \left(|f| + \|f\|_{\mathbb{L}^1}\right), \tag{95}$$

$$|\mathcal{L}_1[f]| \le b_3 \left(|f| + \|f\|_{\mathbb{L}^1}\right), \tag{96}$$

where $b_i,\ i = \overline{1,4}$, are positive constants.

4.1 | Preliminary lemmas

The following lemmas are used to prove Theorem 1.

Lemma 4. *The transformation (17) maps the system (9)–(12) into the following system*

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{97}$$

$$w_t(x,t) = w_{xx}(x,t) - F[w(x,t), w_x(x,t), X(t)], \tag{98}$$

$$w_x(0,t) = 0, \tag{99}$$

$$w(1,t) = 0, \tag{100}$$

where

$$F[w, w_x, X] = \mathcal{K}\left[(\mathcal{L}[w] + \kappa(x)X)\left(w_x + \mathcal{L}_1[w] + \kappa'(x)X\right)\right]. \tag{101}$$

Lemma 5. *There exists $\delta_0 > 0$ such that, if $\|w\|_\infty \le \delta_0$ the functional $F = F[w, w_x, X]$ satisfies*

$$|F| \le c_1\left(|w| + \|w\|_{\mathbb{L}^2}\right)\left(|w_x| + \|w_x\|_{\mathbb{L}^2}\right) + c_2\left(|w|^2 + \|w\|_{\mathbb{L}^2}^2\right) + c_3\left(|w_x| + \|w_x\|_{\mathbb{L}^2}\right)|X(t)| + c_4|X(t)|^2, \tag{102}$$

where $c_i,\ i = \overline{1,4}$, are positive constants.

Proofs for Lemmas (4) and (5) are given in the Appendix.

4.2 | Proof of Theorem 1

Utilizing Lemmas 4 and 5, we can now prove Theorem 1. The proof is divided into four steps. The first and the second step are to analyze the growth of $|X(t)|^2 + \|w(\cdot,t)\|^2_{\mathbb{L}^2}$ and $|X(t)|^2 + \|w_t(\cdot,t)\|^2_{\mathbb{L}^2}$, respectively. The results will be used to proof the stability in the third step.

4.2.1 | Analyzing the growth of $|X(t)|^2 + \|w(\cdot,t)\|^2_{\mathbb{L}^2}$

Consider the following Lyapunov functional for the transformed system (97)–(100)

$$V_3(t) = X(t)^{\intercal} P X(t) + \frac{a}{2}\int_0^1 w^2(x,t)\,\mathrm{d}x, \tag{103}$$

where $P = P^{\intercal} > 0$ is the unique solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^{\intercal} P = -Q, \tag{104}$$

for some positive definite matrix Q. Computing the time derivative of $V_3(t)$, we have

$$\dot V_3(t) = -X(t)^{\intercal} Q X(t) + 2X(t)^{\intercal} PBw(0,t) - a\int_0^1 w_x^2(x,t)\,\mathrm{d}x - a\int_0^1 w(x,t)F[w(x,t),w_x(x,t),X(t)]\,\mathrm{d}x. \tag{105}$$

Using Young's inequality, the second term can be estimated as follows

$$2X(t)^{\intercal} PBw(0,t) \le \frac{\lambda_{\min}(Q)}{2}|X(t)|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)} w(0,t)^2. \tag{106}$$

The last term can be analyzed as follows

$$\left|\int_0^1 w(x,t)F[w(x,t),w_x(x,t),X(t)]\,\mathrm{d}x\right| \le d_1\int_0^1 |w(x,t)||F[w(x,t),w_x(x,t),X(t)]|\,\mathrm{d}x, \tag{107}$$

where $d_1 > 0$. From Lemma 5, there exists a constant $\delta_1 > 0$, such that for $\|w\|_\infty < \delta_1$, it holds that

$$\begin{aligned}\int_0^1 |w(x,t)||F[w(x,t),w_x(x,t),X(t)]|\,\mathrm{d}x &\le d_2\left(\|w_x\|_\infty\|w\|^2_{\mathbb{L}^2} + \|w\|_\infty\|w\|^2_{\mathbb{L}^2} + \|w_x\|_\infty\|w\|_{\mathbb{L}^2}|X(t)| + \|w\|_\infty|X(t)|^2\right)\\ &\le d_3\left(\|w_x\|_\infty\left(\|w\|^2_{\mathbb{L}^2} + |X(t)|^2\right) + \|w\|_\infty\left(\|w\|^2_{\mathbb{L}^2} + |X(t)|^2\right)\right)\\ &\le d_4\left(\|w_x\|_\infty V_3(t) + V_3(t)^{\frac{3}{2}}\right),\end{aligned} \tag{108}$$

where $d_i,\ i=\overline{2,4}$ are positive constants. Here, the second line uses (6) and (102), and the third line is obtained using Young's inequality, that is,

$$\|w\|_{\mathbb{L}^2}|X(t)| \le \frac{1}{2}\|w\|^2_{\mathbb{L}^2} + \frac{1}{2}|X(t)|^2. \tag{109}$$

The last line is obtained from

$$\|w\|_\infty \le d_5\left(\|w_x\|_\infty + V_3(t)^{\frac{1}{2}}\right), \tag{110}$$

for a positive constant d_5, which follows from (6) and (7). Thus, we have

$$\dot{V}_3(t) \leq -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \left(a - \frac{8|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2_{\mathbb{L}^2} + e_1\left(\|w_x\|_\infty V_3(t) + V_3(t)^{\frac{3}{2}}\right), \tag{111}$$

for $e_1 > 0$. Furthermore, for $a > \frac{8|PB|^2}{\lambda_{\min}(Q)} + 2$, we have

$$\dot{V}_3(t) \leq -\lambda_1 V_3(t) + C_1\left(\|w_x\|_\infty V_3(t) + V_3(t)^{\frac{3}{2}}\right), \tag{112}$$

where

$$\lambda_1 = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{2} - \frac{4|PB|^2}{a\lambda_{\min}(Q)}\right\} > 0, \tag{113}$$

and $C_1 > 0$. Note that the right hand side of (112) contains $\|w_x\|_\infty$. According to (8), this term is bounded by $\|w_{xx}\|_{\mathbb{L}^2}$, thus higher regularity is needed. In what follows, we show a relation between $\|w_t\|_{\mathbb{L}^2}$ and $\|w_{xx}\|_{\mathbb{L}^2}$ under the assumption that $\|w\|_\infty$ is small enough.

4.2.2 | Analyzing the growth of $|X(t)|^2 + \|w_t(\cdot,t)\|^2_{\mathbb{L}^2}$

Let us denote $v = w_t$ and $Y = \dot{X}$. Differentiating (97)–(100) with respect to t, we get

$$\dot{Y}(t) = (A + BK)\,Y(t) + Bv(0,t), \tag{114}$$

$$v_t(x,t) = v_{xx}(x,t) - (\mathcal{L}[w] + \kappa(x)X(t))\,v_x - F_1[w(x,t), w_x(x,t), v(x,t), X(t), Y(t)], \tag{115}$$

$$v_x(0,t) = 0, \tag{116}$$

$$v(1,t) = 0, \tag{117}$$

where

$$\begin{aligned} F_1[w(x,t), w_x(x,t), v(x,t), X(t), Y(t)] = {} & \mathcal{K}_1[(\mathcal{L}[w] + \kappa(x)X(t))\,v] + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,w_x] \\ & + q(x,0)\,(\mathcal{L}[w(0,t)] + \kappa(0)X(t))\,v(0,t) + \int_0^x q(x,y)\left(\mathcal{L}_y[w] + \kappa'(y)X(t)\right)v\,\mathrm{d}y \\ & + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,\mathcal{L}_1[w]] + \mathcal{K}[(\mathcal{L}[w] + \kappa(x)X(t))\,\mathcal{L}_1[v]] \\ & + \mathcal{K}[(\mathcal{L}[v] + \kappa(x)Y(t))\,\kappa'(x)X(t)] + \mathcal{K}[(\mathcal{L}[w] + \kappa(x)X(t))\,\kappa'(x)Y(t)]. \end{aligned} \tag{118}$$

Similar to the proof of Lemma 5, this functional can be estimated as follows

$$\begin{aligned} |F_1| \leq {} & f_1(|w| + \|w\|_{\mathbb{L}^2})(|v| + \|v\|_{\mathbb{L}^2}) + f_2(|w_x| + \|w_x\|_{\mathbb{L}^2})(|v| + \|v\|_{\mathbb{L}^2}) \\ & + f_3(|w(0,t)||v(0,t)| + |X||v(0,t)|) + f_4(|v| + \|v\|_{\mathbb{L}^2})|X| + f_5(|w_x| + \|w_x\|_{\mathbb{L}^2})|Y| \\ & + f_6(|w| + \|w\|_{\mathbb{L}^2})|Y| + f_7|X||Y|, \end{aligned} \tag{119}$$

where $f_i,\ i = \overline{1,7}$ are positive constants. Consider the following Lyapunov functional

$$V_4(t) = Y(t)^\intercal PY(t) + \frac{a}{2}\int_0^1 v^2(x,t)\,\mathrm{d}x. \tag{120}$$

Computing its first order partial derivative with respect to time, we have

$$\dot{V}_4(t) = -Y(t)^\intercal QY(t) + 2Y(t)^\intercal PBv(0,t) - a\int_0^1 v_x^2(x,t)\,\mathrm{d}x + \frac{a}{2}\left(\mathcal{L}[w(0,t)] + \kappa(0)X(t)\right)v^2(0,t)$$
$$+\frac{a}{2}\int_0^1 \left(\mathcal{L}_x[w] + \kappa'(x)X(t)\right)v^2\,\mathrm{d}x - a\int_0^1 v(x,t)F_1[w,w_x,v,X(t),Y(t)]\,\mathrm{d}x. \tag{121}$$

Since κ' is bounded and from (96), the fifth term of the right hand side can be estimated as follows

$$\left|\int_0^1 \left(\mathcal{L}_x[w] + \kappa'(x)X(t)\right)v^2\,\mathrm{d}x\right| \le \int_0^1 \left|\left(w_x + \mathcal{L}_1[w] + \kappa'(x)X(t)\right)\right| |v|^2\,\mathrm{d}x$$
$$\le d_6\int_0^1 \left(|w_x| + |w| + \|w\|_{\mathbb{L}^2} + |X(t)|\right)|v|^2\,\mathrm{d}x$$
$$\le d_7\left(\|w_x\|_\infty V_4(t) + V_3(t)^{\frac{1}{2}}V_4(t)\right), \tag{122}$$

for some positive constants $d_i,\ i=\overline{6,7}$. Thus, proceeding similarly with calculation for $\dot{V}_3(t)$, we have from (119)–(122) that

$$\dot{V}_4(t) \le -\lambda_2 V_4(t) + C_2\left(\|w_x\|_\infty V_4(t) + V_3(t)^{\frac{1}{2}}V_4(t)\right). \tag{123}$$

Now, from (98), we have

$$v(x,t) = w_{xx}(x,t) - F[w(x,t), w_x(x,t), X(t)]. \tag{124}$$

From (102), we compute the $\mathbb{L}^2$ norm of w_{xx} as follows

$$\|w_{xx}(x,t)\|_{\mathbb{L}^2} \le \|v(x,t)\|_{\mathbb{L}^2} + \|F[w(x,t),w_x(x,t),X(t)]\|_{\mathbb{L}^2}$$
$$\le \|v(x,t)\|_{\mathbb{L}^2} + d_8\|w\|_\infty\|w_x(x,t)\|_{\mathbb{L}^2} + d_9|X(t)|\|w_x(x,t)\|_{\mathbb{L}^2} + d_{10}|X(t)|^2, \tag{125}$$

where $d_i,\ i=\overline{8,10}$ are positive constants. From (8), we know that $\|w_x(x,t)\|_{\mathbb{L}^2}$ is bounded by $\|w_{xx}(x,t)\|_{\mathbb{L}^2}$. Thus, if we choose $\|w\|_\infty < \min\left\{\delta_1, \frac{1}{2d_8}\right\}$ and $|X(t)| < \min\left\{\frac{1}{2d_9}, \sqrt{\frac{1}{d_{10}}}\right\}$, that is, $\|w\|_\infty$ and $|X(t)|$ are sufficiently small enough, we have

$$\|w_{xx}(x,t)\|_{\mathbb{L}^2} \le d_{11}\|v(x,t)\|_{\mathbb{L}^2}, \tag{126}$$

where $d_{11} > 0$ is a sufficiently large number. Therefore, from (8), (126), and the definition of $V_4(t)$, we have $\|w_x\|_\infty \le d_{12}V_4(t)^{\frac{1}{2}}$ for $d_{12} > 0$.

4.2.3 | Proof of stability

Let us define $S(t) = V_3(t) + V_4(t)$. From (112) and (123), we have

$$\dot{S}(t) \le -\lambda S(t) + CS(t)^{\frac{3}{2}}, \tag{127}$$

for some positive λ and C. Then, for any λ_0 such that $0 < \lambda_0 < \lambda$, we have

$$CS(t)^{\frac{3}{2}} \le (\lambda - \lambda_0)\, S(t),\ \forall S(t) \le \left(\frac{\lambda - \lambda_0}{C}\right)^2, \tag{128}$$

which implies that

$$\dot{S}(t) \le -\lambda_0 S(t),\ \forall S(t) \le \left(\frac{\lambda - \lambda_0}{C}\right)^2. \tag{129}$$

Then, we have $S(t) \to 0$ exponentially. Since $|\dot{X}|^2 \le M\left(|X|^2 + \|w_{xx}\|_{\mathbb{L}^2}\right)$ for some $M > 0$ and $S(t)$ is equivalent to $|X|^2 + \|w_{xx}\|_{\mathbb{L}^2}$ when $\|w\|_\infty$ and $|X|$ are sufficiently small, and the norm $|X|^2 + \|w_{xx}\|_{\mathbb{L}^2}$ is equivalent to the norm $|X(t)|^2 + \|w(\cdot, t)\|_{\mathbb{H}^2}$, then the (w, X)-system is locally exponentially stable in the sense of the norm

$$|X(t)|^2 + \|w(\cdot, t)\|^2_{\mathbb{H}^2}. \tag{130}$$

We note that the (w, X) system and the (u, X) system are equivalent through the backstepping transformation (17).

4.2.4 | Proof of existence of the closed-loop solution

There are extensive results regarding the existence of a solution for the coupled Burgers' PDE-ODE system (85)–(87), for example,.[54,55] Since we assume the pair (A, B) is stabilizable, the forcing term $CX(t)$ is bounded, that is, $CX(t) \in \mathbb{L}^\infty(0, \infty)$. Following theorem 3.1 in Reference 56, if we further assume the initial condition $u_0(x) \in \mathbb{L}^\infty$, then there exists a unique solution that satisfy the coupled Burgers' PDE-ODE system (85)–(87).

5 | OBSERVER DESIGN OF THE COUPLED BURGERS' PDE-ODE SYSTEMS

Let us assume that only $u(0, t)$ is available for measurement. An anticollocated Luenberger observer is designed as follows

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(0, t) + P_0\tilde{u}(0, t), \tag{131}$$

$$\hat{u}_t(x, t) = \hat{u}_{xx}(x, t) - \hat{u}(x, t)\hat{u}_x(x, t) + C\hat{X}(t) + p_1(x)\tilde{u}(0, t), \tag{132}$$

$$\hat{u}_x(0, t) = 0, \tag{133}$$

$$\hat{u}(1, t) = U(t), \tag{134}$$

where $p_1(x) = \Theta(x)P_0$ and P_0 is chosen such that $A - P_0\Theta(0)$ is Hurwitz. Let $\tilde{X} = X - \hat{X}$ and $\tilde{u} = u - \hat{u}$, the error equation is given by

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}(0, t), \tag{135}$$

$$\tilde{u}_t(x, t) = \tilde{u}_{xx}(x, t) - u(x, t)u_x(x, t) + \hat{u}(x, t)\hat{u}_x(x, t) + C\tilde{X}(t) - p_1(x)\tilde{u}(0, t), \tag{136}$$

$$\tilde{u}_x(0, t) = 0, \tag{137}$$

$$\tilde{u}(1, t) = 0. \tag{138}$$

The exponential stability of this error system is proved in Section 6. We define the following pair of direct and inverse transformations

$$\hat{w}(x, t) = \hat{u}(x, t) - \int_0^x q(x, y)\hat{u}(y, t)\, dy - \gamma(x)\hat{X}(t), \tag{139}$$

$$\hat{u}(x,t)=\hat{w}(x,t)+\int_0^x p(x,y)\hat{w}(y,t)\,\mathrm{d}y+\kappa(x)\hat{X}(t), \tag{140}$$

then the following lemma holds for the resulting transformed $(\tilde{w},\tilde{X})$-system.

Lemma 6. *If the observer gain is given by $p_1(x)=\Theta(x)P_0$, where P_0 is chosen such that $A-P_0\Theta(0)$ is Hurwitz and $\Theta(x)$ is given by (67), then the transformations (59) and (140) map (135)–(138) into the following system*

$$\dot{\tilde{X}}(t)=(A-P_0\Theta(0))\tilde{X}(t)-P_0\tilde{w}(0,t), \tag{141}$$

$$\tilde{w}_t(x,t)=\tilde{w}_{xx}(x,t)-H[\hat{w},\hat{w}_x,\hat{X}(t),\tilde{w},\tilde{w}_x,\tilde{X}(t)], \tag{142}$$

$$\tilde{w}_x(0,t)=0, \tag{143}$$

$$\tilde{w}(1,t)=0, \tag{144}$$

where

$$\begin{aligned}H[\hat{w}(x,t),\hat{w}_x(x,t),\hat{X}(t),\tilde{w}(x,t),\tilde{w}_x(x,t),\tilde{X}(t)]=&\left(\mathcal{L}[\hat{w}(x,t)]+\kappa(x)\hat{X}(t)\right)\left(\tilde{w}_x(x,t)+\Theta'(x)\tilde{X}(t)\right)\\&+\left(\tilde{w}(x,t)+\Theta(x)\tilde{X}(t)\right)(\hat{w}_x(x,t)+\mathcal{L}_1[\hat{w}(x,t)]\\&+\kappa'(x)\hat{X}(t)+\tilde{w}_x(x,t)+\Theta'(x)\tilde{X}(t)\right).\end{aligned} \tag{145}$$

Lemma 6 is proved in the Appendix. The stability of the error system (141)–(144) is proved simultaneously with the output feedback control given in the next section. The proof requires an estimate for the functional H, which is given in Lemma 7. Lemma 7 is proved in the Appendix.

Lemma 7. *There exists a positive constant δ_2 such that, if $\|\hat{w}\|_\infty+\|\tilde{w}\|_\infty<\delta_2$ then the functional H satisfies*

$$\begin{aligned}|H|\le{}& s_1(|\hat{w}|+\|\hat{w}\|_{\mathbb{L}^2})(|\tilde{w}_x|+\|\tilde{w}_x\|_{\mathbb{L}^2})+s_2(|\hat{w}|+\|\hat{w}\|_{\mathbb{L}^2})|\tilde{X}(t)|\\&+s_3(|\tilde{w}_x|+\|\tilde{w}_x\|_{\mathbb{L}^2})|\hat{X}(t)|+s_4|\hat{X}(t)||\tilde{X}(t)|+s_5(|\tilde{w}|+\|\tilde{w}\|_{\mathbb{L}^2})(|\hat{w}_x|+\|\hat{w}_x\|_{\mathbb{L}^2})\\&+s_6(|\tilde{w}|+\|\tilde{w}\|_{\mathbb{L}^2})(|\hat{w}|+\|\hat{w}\|_{\mathbb{L}^2})+s_7(|\tilde{w}|+\|\tilde{w}\|_{\mathbb{L}^2})|\hat{X}(t)|\\&+s_8(|\hat{w}_x|+\|\hat{w}_x\|_{\mathbb{L}^2})|\tilde{X}(t)|+s_9(|\hat{w}|+\|\hat{w}\|_{\mathbb{L}^2})|\tilde{X}(t)|+s_{10}|\hat{X}(t)||\tilde{X}(t)|\\&+s_{11}(|\tilde{w}|+\|\tilde{w}\|_{\mathbb{L}^2})(|\tilde{w}_x|+\|\tilde{w}_x\|_{\mathbb{L}^2})+s_{12}(|\tilde{w}|+\|\tilde{w}\|_{\mathbb{L}^2})|\tilde{X}(t)|+s_{13}(|\tilde{w}_x|+\|\tilde{w}_x\|_{\mathbb{L}^2})|\tilde{X}(t)|+s_{14}|\tilde{X}(t)|^2,\end{aligned} \tag{146}$$

where $s_i,\ i=\overline{1,14}$ are positive constants.

6 | OUTPUT FEEDBACK CONTROL OF THE COUPLED BURGERS' PDE-ODE SYSTEMS

To proof the output feedback stabilization problem, first we consider a transformed system of the observer equation (131)–(134) as follows.

Lemma 8. *The transformations (59) and (139) map (131)–(134) into the following system*

$$\dot{\hat{X}}(t)=(A+BK)\hat{X}(t)+B\hat{w}(0,t)+(B+P_0)\left(\tilde{w}(0,t)+\Theta(0)\tilde{X}(t)\right), \tag{147}$$

$$\hat{w}_t(x,t)=\hat{w}_{xx}(x,t)-\hat{F}[\hat{w}(x,t),\hat{w}_x(x,t),\hat{X}(t)]+\overline{p}_1(x)\left(\tilde{w}(0,t)+\Theta(0)\tilde{X}(t)\right), \tag{148}$$

$$\hat{w}_x(0,t)=0, \tag{149}$$

$$\hat{w}(1,t)=0, \tag{150}$$

where $\overline{p}_1(x)$ is given by (78), and

$$\hat{F}[\hat{w},\hat{w}_x,\hat{X}] = \mathcal{K}\left[\left(\mathcal{L}[\hat{w}]+\kappa(x)\hat{X}\right)\left(\hat{w}_x+\mathcal{L}_1[\hat{w}]+\kappa'(x)\hat{X}\right)\right]. \tag{151}$$

Lemma 8 is proved in the Appendix. Replacing the state in (47) with its estimate $(\hat{u},\hat{X})$ obtained from the designed observer (131)–(134), an output feedback control law is given by (79) for which the following theorem holds.

Theorem 2. *Assume that A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k\in\mathbb{N}$. For any initial data $X(0),\hat{X}(0)\in\mathbb{R}^n$ and $u_0(x),\hat{u}_0(x)\in\mathbb{H}^2([0,1])$ compatible with the control law (79), there exists $\delta>0$ such that if $\|u_0\|_{\mathbb{H}^2}+\|\hat{u}_0\|_{\mathbb{H}^2}+|X(0)|+|\hat{X}(0)|\le\delta$, then the closed-loop system consisting of the plant (9)–(12), the controller (79), and the observer (131)–(134) has a unique classical solution and is exponentially stabilized in the sense of the norm*

$$\left\|\left(X,u,\hat{X},\hat{u}\right)\right\|^2 = |X|^2+\|u\|_{\mathbb{H}^2}^2+\left|\hat{X}\right|^2+\|\hat{u}\|_{\mathbb{H}^2}^2. \tag{152}$$

Proof. First, for system (141)–(144) and (147)–(150), we employ the following Lyapunov functional

$$U_1(t)=\hat{X}(t)^\intercal\hat{P}\hat{X}(t)+\frac{\hat{a}}{2}\int_0^1\hat{w}^2(x,t)\,\mathrm{d}x+\overline{d}\left(\tilde{X}(t)^\intercal\tilde{P}\tilde{X}(t)+\frac{\tilde{a}}{2}\int_0^1\tilde{w}^2(x,t)\,\mathrm{d}x\right), \tag{153}$$

where the matrices $\hat{P}=\hat{P}^\intercal>0$ and $\tilde{P}=\tilde{P}^\intercal>0$ are the solutions to the Lyapunov functionals (81) and (82), respectively. The parameters $\hat{a}$, $\tilde{a}$, and $\overline{d}$ are to be determined later. Computing the first derivative of U_1 with respect to t, we have

$$\begin{aligned}\dot{U}_1(t)=&-\hat{X}(t)^\intercal\hat{Q}\hat{X}(t)+2\hat{X}(t)^\intercal\hat{P}\left(B\hat{w}(0,t)+(B+P_0)\left(\tilde{w}(0,t)+\Theta(0)\tilde{X}(t)\right)\right)-\hat{a}\int_0^1\hat{w}_x^2(x,t)\,\mathrm{d}x\\&+\hat{a}\rho\int_0^1\hat{w}(x,t)\left(\tilde{w}(0,t)+\Theta(0)\tilde{X}(t)\right)\,\mathrm{d}x-\hat{a}\int_0^1\hat{w}(x,t)\hat{F}[\hat{w}(x,t),\hat{w}_x(x,t),\hat{X}(t)]\,\mathrm{d}x\\&-\overline{d}\tilde{X}(t)^\intercal Q\tilde{X}(t)-2\overline{d}\tilde{X}(t)^\intercal PP_0\tilde{w}(0,t)-\overline{d}\tilde{a}\int_0^1\tilde{w}_x^2(x,t)\,\mathrm{d}x-\overline{d}\tilde{a}\int_0^1\tilde{w}(x,t)H[\hat{w},\hat{w}_x,\hat{X},\tilde{w},\tilde{w}_x,\tilde{X}]\,\mathrm{d}x,\end{aligned} \tag{154}$$

where $\rho=\max_{x\in[0,1]}\overline{p}_1(x)$. Following the proof for Theorem 1, we consider a positive constant δ_2 that satisfies Lemma 7. If we assume that $\|\hat{w}\|_\infty+\|\tilde{w}\|_\infty<\delta_2$, then the fifth and ninth terms on the right hand side of (154) can be estimated as follows

$$\left|\int_0^1\hat{w}(x,t)\hat{F}[\hat{w}(x,t),\hat{w}_x(x,t),\hat{X}(t)]\,\mathrm{d}x\right|\le r_1\left(\|\hat{w}_x\|_\infty\left(\|\hat{w}\|_{\mathbb{L}^2}^2+|\hat{X}(t)|^2\right)+\|\hat{w}\|_\infty\left(\|\hat{w}\|_{\mathbb{L}^2}^2+|\hat{X}(t)|^2\right)\right), \tag{155}$$

$$\begin{aligned}\left|\int_0^1\tilde{w}(x,t)H[\hat{w},\hat{w}_x,\hat{X},\tilde{w},\tilde{w}_x,\tilde{X}]\,\mathrm{d}x\right|\le\;& r_2\left(\|\tilde{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}\|\hat{w}\|_{\mathbb{L}^2}+\|\tilde{w}\|_{\mathbb{L}^2}\|\hat{w}\|_{\mathbb{L}^2}|\tilde{X}(t)|+\|\tilde{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}|\hat{X}(t)|\right)\\&+r_3\left(\|\tilde{w}\|_{\mathbb{L}^2}|\hat{X}(t)||\tilde{X}(t)|+\|\hat{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}^2+\|\hat{w}\|_{\mathbb{L}^2}\|\tilde{w}\|_{\mathbb{L}^2}^2\right)\\&+r_4\left(\|\tilde{w}\|_{\mathbb{L}^2}^2|\hat{X}(t)|+\|\hat{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}|\hat{X}(t)|+\|\hat{w}\|_{\mathbb{L}^2}\|\tilde{w}\|_{\mathbb{L}^2}|\tilde{X}(t)|\right)\\&+r_5\left(\|\tilde{w}\|_{\mathbb{L}^2}|\hat{X}(t)||\tilde{X}(t)|+\|\tilde{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}^2+\|\tilde{w}\|_{\mathbb{L}^2}^2|\tilde{X}(t)|\right)\\&+r_6\left(\|\tilde{w}_x\|_\infty\|\tilde{w}\|_{\mathbb{L}^2}|\tilde{X}(t)|+\|\tilde{w}\|_{\mathbb{L}^2}^2|\tilde{X}(t)|^2\right),\end{aligned} \tag{156}$$

where $r_i,\ i=\overline{1,6}$ are positive constants. Thus, using Poincare's, Agmon's, and Young's inequalities, (154) can be estimated as follows

$$\dot{U}_1(t)\le-\left(\frac{\lambda_{\min}(\hat{Q})}{2}-\overline{c}|\hat{P}(B+P_0)|^2\right)|\hat{X}(t)|^2-\left(\frac{\hat{a}}{2}-\frac{1}{2}-16\frac{|\hat{P}B|^2}{\lambda_{\min}(\hat{Q})}\right)\|\hat{w}_x\|^2$$

$$
- \left(\frac{\lambda_{\min}(\tilde{Q})}{2}\overline{d} - \left(\frac{1}{\overline{c}} + 4\hat{a}\rho^2 \right) |\Theta(0)|^2 \right) |\tilde{X}(t)|^2 - \overline{d} \left(\tilde{a} - 8\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} \right) \|\tilde{w}_x\|^2
$$

$$
+ \left(16\frac{|\hat{P}(B+P_0)|^2}{\lambda_{\min}(\hat{Q})} + 16\hat{a}\rho^2 \right) \|\tilde{w}_x\|^2 + U_1(t)^{\frac{3}{2}} + (\|\tilde{w}_x\|_\infty + \|\hat{w}_x\|_\infty)\, U_1(t), \tag{157}
$$

where $0 < \overline{c} < \frac{\lambda_{\min}(\hat{Q})}{2|\hat{P}(B+P_0)|^2}$. Furthermore, we can choose successively

$$
\hat{a} > 32\frac{|\hat{P}B|^2}{\lambda_{\min}(\hat{Q})} + 1, \tag{158}
$$

$$
\overline{d} > \frac{2}{\lambda_{\min}(\tilde{Q})} \left(\frac{1}{\overline{c}} + 4\hat{a}\rho^2 \right) |\Theta(0)|^2, \tag{159}
$$

$$
\tilde{a} > 8\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} + \frac{1}{\overline{d}} \left(16\frac{|\hat{P}(B+P_0)|^2}{\lambda_{\min}(\hat{Q})} - 16\hat{a}\rho^2 \right), \tag{160}
$$

such that

$$
\dot{U}_1(t) \leq -\lambda_3 U_1(t) + C_3 \left((\|\tilde{w}_x\|_\infty + \|\hat{w}_x\|_\infty)\, U_1(t) + U_1(t)^{\frac{3}{2}} \right).
$$

Let us denote $\hat{v} = \hat{w}_t$, $\hat{Y} = \dot{\hat{X}}$, $\tilde{v} = \tilde{w}_t$, $\tilde{Y} = \dot{\tilde{X}}$. Differentiating (141)–(144) with respect to t, we have

$$
\dot{\tilde{Y}}(t) = (A - P_0\Theta(0))\, \tilde{Y}(t) - P_0\tilde{v}(0,t), \tag{161}
$$

$$
\begin{aligned}
&\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) - H_1[\hat{w}(x,t), \hat{w}_x, \hat{v}, \hat{X}, \hat{Y}, \tilde{w}, \tilde{w}_x, \tilde{v}, \tilde{X}, \tilde{Y}], \\
&\tilde{v}_x(0,t) = 0,
\end{aligned} \tag{162}
$$

$$
\tilde{v}(1,t) = 0. \tag{163}
$$

Furthermore, differentiating (147)–(150) with respect to t, we have

$$
\dot{\hat{Y}}(t) = (A + BK)\, \hat{Y}(t) + B\hat{v}(0,t) + (B + P_0) \left(\tilde{v}(0,t) + \Theta(0)\tilde{Y}(t) \right), \tag{164}
$$

$$
\hat{v}_t(x,t) = \hat{v}_{xx}(x,t) - \left(\mathcal{L}[\hat{w}] + \kappa(x)\hat{X}(t) \right) \hat{v}_x - \hat{F}_1[\hat{w}(x,t), \hat{w}_x(x,t), \hat{v}(x,t), \hat{X}(t), \hat{Y}(t)] + \overline{p}_1(x) \left(\tilde{v}(0,t) + \Theta(0)\tilde{Y}(t) \right), \tag{165}
$$

$$
\hat{v}_x(0,t) = 0, \tag{166}
$$

$$
\hat{v}(1,t) = 0. \tag{167}
$$

Here we denote $H_1 = H_t$ and $\hat{F}_1 = \hat{F}_t$, respectively. We employ the second Lyapunov functional

$$
U_2(t) = \hat{Y}(t)^\intercal \hat{P}\hat{Y}(t) + \frac{\hat{a}}{2}\int_0^1 \hat{v}^2(x,t)\, \mathrm{d}x + \overline{d} \left(\tilde{Y}(t)^\intercal \tilde{P}\tilde{Y}(t) + \frac{\tilde{a}}{2}\int_0^1 \tilde{v}^2(x,t)\, \mathrm{d}x \right). \tag{168}
$$

Following the same argument as in Section 4.2 step 2, the first derivative of U_2 with respect to t is given by

$$
\dot{U}_2(t) \leq -\lambda_4 U_2(t) + C_4 \left((\|\tilde{w}_x\|_\infty + \|\hat{w}_x\|_\infty)\, U_2(t) + U_1(t)^{\frac{1}{2}} U_2(t) \right).
$$

Defining $T(t) = U_1(t) + U_2(t)$, and following the same argument as in Section 4.2.3, completes the proof. ■

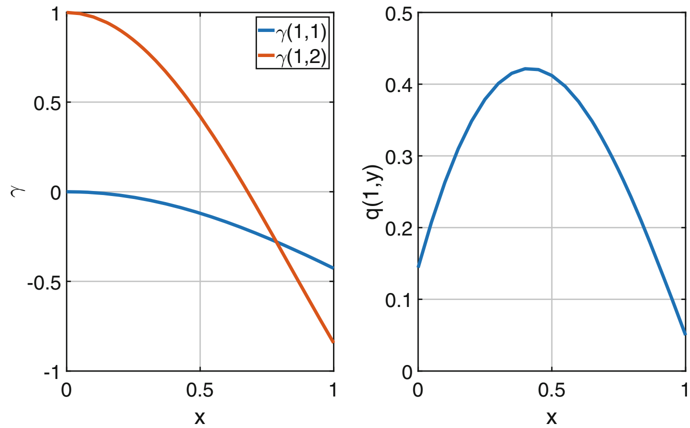
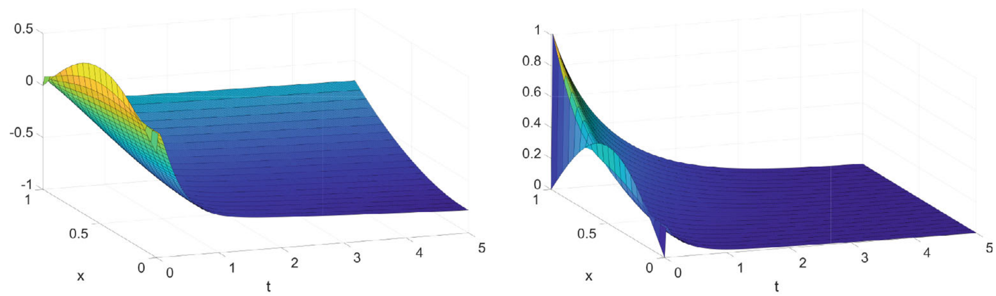


FIGURE 2 Value of the control gains



(A) Uncontrolled (B.) Controlled

FIGURE 3 Uncontrolled and controlled case using state feedback

7 | NUMERICAL SIMULATIONS

7.1 | Control law performance

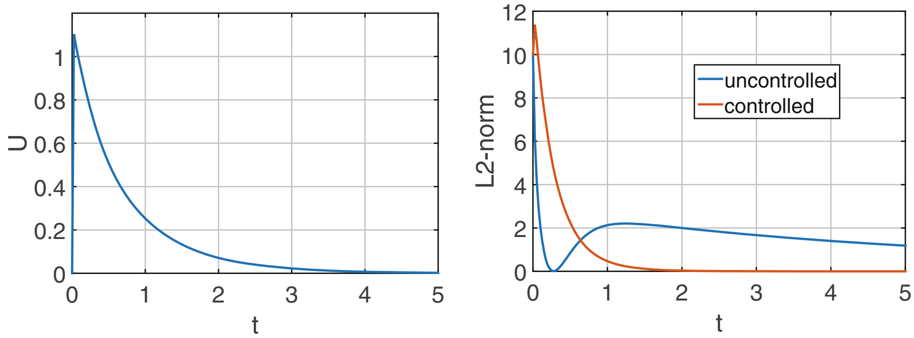
Consider the coupled Burgers' PDE-ODE system (85)–(87). In this simulation, we use the following data

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -4 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{and } C = \begin{pmatrix} 1 & 1 \end{pmatrix}. \tag{169}$$

The coupled system is discretized using a finite difference method with $\Delta t = 0.001$ and $\Delta x = 0.05$. The space discretization leads to a state vector of dimension $n = 20$, which is sufficient to guarantee the gains and the solutions to converge. Before evaluating the controller $U(t)$ given by (47), first the control gains $\gamma(1)$ and $q(1, y)$ are calculated using (30) and (34), respectively. The control gains can be seen in Figure 2. Here, the value of $\gamma(1) = \begin{pmatrix} -0,4271 & -0,8433 \end{pmatrix}$. Furthermore, $K = \gamma(0) = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and the eigenvalues of $A + BK$ is -1 and -3.

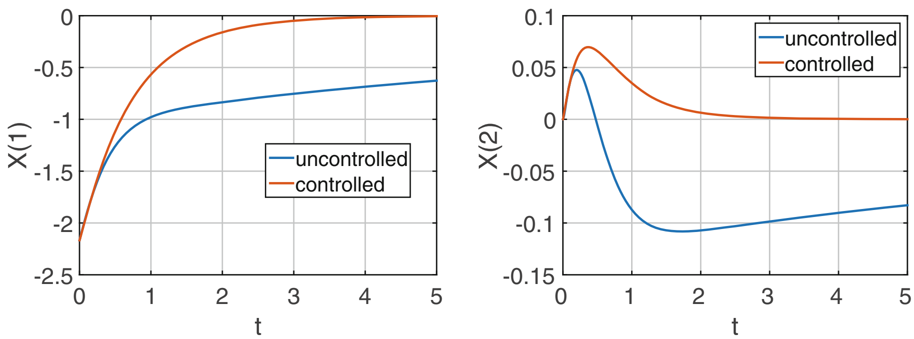
To evaluate the performance of the controller, we compare two cases. In the first case, we set $U(t) = 0$, which means no control applies. In this case, we would expect the system will not converge to its equilibrium. In the second case, we use the controller (47) to stabilize the coupled system. The results can be seen in Figure 3.

There was a significant difference between the two cases. After 5 s, we can observe with zero control input the system have a steady state error. When the controller is used, it compensates the error to zero. The controller could be seen in Figure 4A, while the performance of the controller could be seen from the value of the $\|u\|^2_{\mathbb{L}^2}$-norm of the closed loop system in Figure 4B.



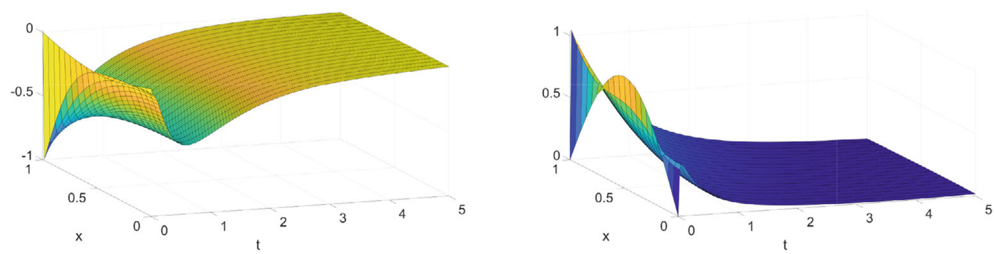
(A) Controller $U(t)$ (B) Comparison of $\|u\|^2_{\mathbb{L}^2}$

FIGURE 4 Control performance



(A) $X(1)$ (B) $X(2)$

FIGURE 5 ODE state $X(t)$



(A) Error $\tilde{u}$ (B) Stabilization using output feedback control

FIGURE 6 Output feedback control

The ODE states can be seen in Figure 5. It can be observed in the controlled case the values go to zero after $t = 5$ s, while in the uncontrolled case the values go to nonzero equilibrium.

Relying on measurement only at $x = 0$, the error between the observer and the real system converges to zero, as can be seen in Figure 6A. The estimation from the observer is used in the controller as an output feedback for the system, as can be seen in Figure 6B.

7.2 | Effect of viscosity coefficient

In this simulation, we set the viscosity coefficient $\epsilon = 1$. If the viscosity coefficient is smaller, that is, $\epsilon = 0.01$, the control law will be affected. In this case, the convergence rate is slower, as can be seen in Figure 7.

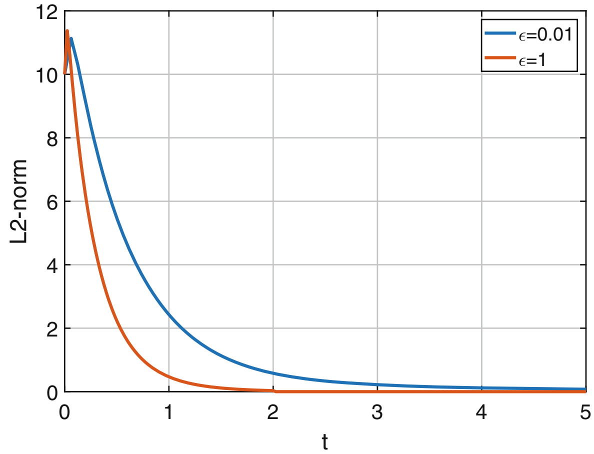
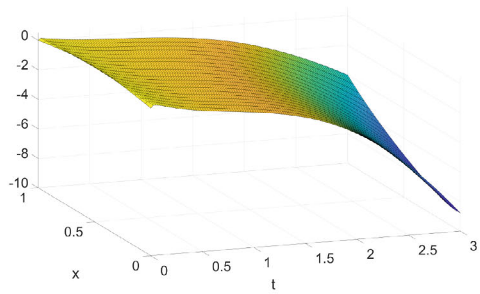
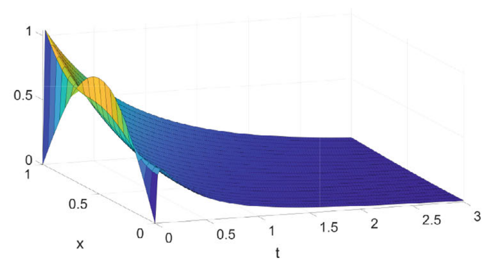


FIGURE 7 Comparison of the control law performance with different viscosity coefficient



(A) Open loop system of Burgers' PDE-ODE system



(B) Stabilization using output feedback control

FIGURE 8 Stabilization of Burgers' PDE-ODE system

7.3 | Control of an unstable system

In the previous examples, the ODE equation is stable, that is, all eigenvalues of A are located in the left half plane. If A is given by

$$A = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \tag{170}$$

then the eigenvalues of A are -1 and 1. In this case, we choose $K = \begin{pmatrix} -0.1 & -1 \end{pmatrix}$, such that $A + BK$ is Hurwitz. Figure 8 shows the comparison between no control setting and output feedback control of the Burgers' PDE-ODE system. Without control, the solution becomes unstable after $t = 3$ s, while using the output feedback controller the solution goes to its equilibrium.

8 | CONCLUSIONS

We present local boundary stabilization of a coupled Burgers' PDE-ODE system with actuation only at one end of the spatial interval. The full-state feedback controller is developed from the stabilization problem of the linearized system, for which $\mathbb{H}^2$ local exponential stability is achieved for the resulting closed-loop coupled Burgers' PDE-ODE system. An observer is developed for the coupled Burgers' PDE-ODE system using only one boundary measurement. Together with the controller, the observer is used in the output feedback regulation. Numerical simulations show the performance of the output feedback controller satisfactorily.

ACKNOWLEDGMENT

The authors would like to thank Dr. Miroslav Krstic and Dr. Marius Tucsnak for their insightful suggestions on a preliminary version of the result in this article.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests for this article.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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How to cite this article: Hasan A, Tang S-X. Boundary control of a coupled Burgers' PDE-ODE system. *Int J Robust Nonlinear Control*. 2022;32(10):5812-5836. doi: 10.1002/rnc.6145

APPENDIX A. PROOF OF THE LEMMAS

A.1 Proof of Lemma 4

Calculating the first and the second order derivatives of (17) with respect to x, we have

$$w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)\,\mathrm{d}y - \gamma'(x)X(t), \tag{A1}$$

$$w_{xx}(x,t) = u_{xx}(x,t) - (q(x,x))'u(x,t) - q(x,x)u_x(x,t) - q_x(x,x)u(x,t) - \int_0^x q_{xx}(x,y)u(y,t)\,\mathrm{d}y - \gamma''(x)X(t). \tag{A2}$$

Furthermore, calculating the first order derivative of (17) with respect to t, we get

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) - u(x,t)u_x(x,t) + CX(t) - q(x,x)u_x(x,t) + q(x,0)u_x(0,t) + q_y(x,x)u(x,t) - q_y(x,0)u(0,t) \\ &- \int_0^x q_{yy}(x,y)u(y,t)\,\mathrm{d}y - \int_0^x q(x,y)CX(t)\,\mathrm{d}y - \gamma(x)\left(AX(t) + Bu(0,t)\right) + \int_0^x q(x,y)u(y,t)u_y(y,t)\,\mathrm{d}y. \end{aligned} \tag{A3}$$

Thus, we have

$$w_t(x,t) = w_{xx}(x,t) - u(x,t)u_x(x,t) + \int_0^x q(x,y)u(y,t)u_y(y,t)\,\mathrm{d}y, \tag{A4}$$

where the kernel equations (22)–(27) are used. In view of (90), (92), and the inverse transformation (35), we have

$$u(x,t) = \mathcal{L}[w(x,t)] + \kappa(x)X(t), \tag{A5}$$

$$u_x(x,t) = w_x(x,t) + \mathcal{L}_1[w(x,t)] + \kappa'(x)X(t). \tag{A6}$$

Therefore, from (89), we obtain (101). Furthermore, from (9), (17), and (26), we have

$$\dot{X}(t) = AX(t) + B\left(w(0,t) + KX(t)\right). \tag{A7}$$

The boundary condition (99) is derived from (A1), (15), (23), and (27), and the boundary condition (100) is derived from (17), (12), and (47). Thus, this completes the proof.

A.2 Proof of Lemma 5

The functional F can be written as

$$\begin{aligned} F[w,w_x,X] = {} & \mathcal{K}\left[\mathcal{L}[w]w_x\right] + \mathcal{K}\left[\mathcal{L}[w]\mathcal{L}_1[w]\right] + \mathcal{K}\left[\mathcal{L}[w]\kappa'(x)X\right] + \mathcal{K}\left[\kappa(x)Xw_x\right] \\ & + \mathcal{K}\left[\kappa(x)X\mathcal{L}_1[w]\right] + \mathcal{K}\left[\kappa(x)X\kappa'(x)X\right]. \end{aligned} \tag{A8}$$

The first term, with the help of Cauchy–Schwarz inequality, (94), and the fact that the kernel $q(x,y)$ is bounded, can be estimated as follows

$$\begin{aligned} \left|\mathcal{K}\left[\mathcal{L}[w]w_x\right]\right| &\le |\mathcal{L}[w]w_x| + \left|\int_0^x q(x,y)\mathcal{L}[w]w_y\,\mathrm{d}y\right| \\ &\le |\mathcal{L}[w]w_x| + g_1\sqrt{\int_0^x q(x,y)^2(|w| + \|w\|_{\mathbb{L}^1})^2\,\mathrm{d}y}\sqrt{\int_0^x \|w_y\|^2_{\mathbb{L}^2}\,\mathrm{d}y} \\ &\le g_2\left(|w| + \|w\|_{\mathbb{L}^2}\right)|w_x| + g_3\|w\|_{\mathbb{L}^2}\|w_x\|_{\mathbb{L}^2}, \end{aligned} \tag{A9}$$

where $g_i,\ i = \overline{1,3}$ denote positive constants and (6) is used in the last line. Similarly, the second term can be estimated as follows

$$\left|\mathcal{K}\left[\mathcal{L}[w]\mathcal{L}_1[w]\right]\right| \le |\mathcal{L}[w]\mathcal{L}_1[w]| + \left|\int_0^x q(x,y)\mathcal{L}[w]\mathcal{L}_1[w]\,\mathrm{d}y\right|$$

$$
\begin{aligned}
&\leq |\mathcal{L}[w]\mathcal{L}_1[w]| + g_4\sqrt{\int_0^x q(x,y)^2(|w| + \|w\|_{\mathbb{L}^1})^4 \, dy} \\
&\leq g_5(|w| + \|w\|_{\mathbb{L}^2})^2 + g_6\|w\|_{\mathbb{L}^2}^2,
\end{aligned} \tag{A10}
$$

where $g_i, \ i = \overline{4,6}$ are suitable positive constants. Since $\kappa(x)$ and $\kappa'(x)$ are bounded, the third term is estimated as

$$
\begin{aligned}
|\mathcal{K}\left[\mathcal{L}[w]\kappa'(x)X\right]| &\leq |\mathcal{L}[w]\kappa'(x)X| + \left|\int_0^x q(x,y)\mathcal{L}[w]\kappa'(y)X \, dy\right| \\
&\leq |\mathcal{L}[w]\kappa'(x)X| + g_7|X|\sqrt{\int_0^x q^2(|w| + \|w\|_{\mathbb{L}^1})^2 \, dy} \\
&\leq g_8\left(|w| + \|w\|_{\mathbb{L}^2}\right)|X| + g_9|X|\|w\|_{\mathbb{L}^2},
\end{aligned} \tag{A11}
$$

where $g_i, \ i = \overline{7,9}$ are positive constants. The forth term can be estimated as

$$
\begin{aligned}
|\mathcal{K}\left[\kappa(x)Xw_x\right]| &\leq |\kappa(x)Xw_x| + \left|\int_0^x q(x,y)\kappa(y)Xw_y \, dy\right| \\
&\leq |\kappa(x)Xw_x| + \sqrt{\int_0^x q(x,y)^2\kappa(y)^2|X|^2 \, dy}\sqrt{\int_0^x w_y^2 \, dy} \\
&\leq g_{10}|X||w_x| + g_{11}|X|\|w_x\|_{\mathbb{L}^2},
\end{aligned} \tag{A12}
$$

where $g_i, \ i = \overline{10,11}$ denote positive constants. The fifth term is estimated similarly to the third term. Finally the last term can be estimated as follows

$$
\begin{aligned}
|\mathcal{K}\left[\kappa(x)X\kappa'(x)X\right]| &\leq |\kappa(x)X\kappa'(x)X| + \left|\int_0^x q(x,y)\kappa(y)X\kappa'(y)X \, dy\right| \\
&\leq |\kappa(x)X\kappa'(x)X| + \sqrt{\int_0^x q(x,y)^2\kappa(y)^2|X|^2 \, dy}\sqrt{\int_0^x \kappa'(x)^2|X|^2 \, dy} \\
&\leq g_{12}|X|^2,
\end{aligned} \tag{A13}
$$

where g_{12} denote a positive constant. Adding all terms completes the proof.

A.3 Proof of Lemma 6

Plugging (59) into (135) yields

$$
\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)\right). \tag{A14}
$$

Thus, we have (141). Computing the derivatives of (59) with respect to x and t, we have

$$
\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - \Theta'(x)\tilde{X}(t), \tag{A15}
$$

$$
\tilde{w}_{xx}(x,t) = \tilde{u}_{xx}(x,t) - \Theta''(x)\tilde{X}(t), \tag{A16}
$$

$$
\tilde{w}_t(x,t) = \tilde{u}_{xx}(x,t) - u(x,t)u_x(x,t) + \hat{u}(x,t)\hat{u}_x(x,t) + \left(\Theta(x)P_0 - p_1(x)\right)\tilde{u}(0,t) - \left(\Theta(x)A - C\right)\tilde{X}(t). \tag{A17}
$$

Subtracting (A17) by (A16), we have

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - \hat{u}(x,t)\tilde{u}_x(x,t) - \tilde{u}(x,t)\hat{u}_x(x,t) - \tilde{u}(x,t)\tilde{u}_x(x,t), \tag{A18}$$

where (64) is used. From (140), (94), and (96), we have

$$\hat{u}(x,t) = \mathcal{L}[\hat{w}(x,t)] + \kappa(x)\hat{X}(t), \tag{A19}$$

$$\hat{u}_x(x,t) = \hat{w}_x(x,t) + \mathcal{L}_1[\hat{w}(x,t)] + \kappa'(x)\hat{X}(t). \tag{A20}$$

Substituting the above equations into (A18) yields (142). The boundary conditions (143) and (144) are obtained from (A15) and (59), with the help of (65)–(66), evaluated at $x = 0$ and $x = 1$, respectively.

A.4 Proof of Lemma 7

Lemma 7 is proved similarly to Lemma 5. First, the functionals $\mathcal{L}$ and $\mathcal{L}_1$ are calculated using (90) and (92), respectively. Afterward, using Cauchy–Schwarz inequality, each of the terms can be proven to be bounded.

A.5 Proof of Lemma 8

Plugging (59) and (139) into (131) yields

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\left(\hat{w}(0,t) + KX(t)\right) + (B + P_0)\left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)\right). \tag{A21}$$

Thus, we have (147). Computing the derivatives of (139) with respect to x, we have

$$\hat{w}_x(x,t) = \hat{u}_x(x,t) - q(x,x)\hat{u}(x,t) - \int_0^x q_x(x,y)\hat{u}(y,t)\,\mathrm{d}y - \gamma'(x)\hat{X}(t), \tag{A22}$$

$$\hat{w}_{xx}(x,t) = \hat{u}_{xx}(x,t) - (q(x,x))'\hat{u}(x,t) - q(x,x)\hat{u}_x(x,t) - q_x(x,x)\hat{u}(x,t) - \int_0^x q_{xx}(x,y)\hat{u}(y,t)\,\mathrm{d}y - \gamma''(x)\hat{X}(t). \tag{A23}$$

Furthermore, calculating the first order derivative of (139) with respect to t, we get

$$\begin{aligned}\hat{w}_t(x,t) = {} & \hat{u}_{xx}(x,t) - \hat{u}(x,t)\hat{u}_x(x,t) + C\hat{X}(t) - q(x,x)\hat{u}_x(x,t) + q(x,0)\hat{u}_x(0,t) + q_y(x,x)\hat{u}(x,t) - q_y(x,0)\hat{u}(0,t) \\ & - \int_0^x q_{yy}(x,y)\hat{u}(y,t)\,\mathrm{d}y - \int_0^x q(x,y)C\hat{X}(t)\,\mathrm{d}y - \gamma(x)\left(A\hat{X}(t) + B\hat{u}(0,t)\right) \\ & + \int_0^x q(x,y)\hat{u}(y,t)\hat{u}_y(y,t)\,\mathrm{d}y + \left(p_1(x) - \gamma(x)(B + P_0)\right)\tilde{u}(0,t) + \int_0^x q(x,y)p_1(y)\,\mathrm{d}y\tilde{u}(0,t).\end{aligned} \tag{A24}$$

Arranging the terms and following the same steps as proving Lemma 4 completes the proof.