2015 American Control Conference
Palmer House Hilton
July 1-3, 2015. Chicago, IL, USA

Stabilization of Linearized Korteweg-de Vries Systems with Anti-diffusion by Boundary Feedback with Non-collocated Observation

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***Abstract*— This paper addresses the problem of stabilizing a class of one-dimensional linearized Korteweg-de Vries systems with possible anti-diffusion (LKdVA for short), through control at one end and non-collocated observation at the other end. An exponentially convergent observer is designed, and then a dynamical stabilizing output feedback boundary controller is constructed based on the observer. The resulting closed-loop systems can achieve arbitrary exponential decay rate. In order to derive invertibility of the kernel function in the backstepping transformation between the observer error systems and its corresponding target systems, stabilizing of a critical case of LKdVA is considered in the Appendix, which can also be treated as a preliminary problem for the main part of this paper.**

***Index Terms*— Linearized Korteweg-de Vries systems; Anti-diffusion; Observer; Output feedback; Backstepping.**

I. INTRODUCTION

The Korteweg-de Vries (KdV) equation can be used to model waves on shallow water surfaces and ion-acoustic waves in plasmas, etc. It has thus been intensively studied by many mathematicians and physicists. Controllability and stabilization of the Korteweg-de Vries (KdV for short) systems have been topics of active research (see, e.g., [1], [2]). In [3], the authors use PDE backstepping method to stabilize a linearized KdV system.

When anti-diffusion exists in systems such as Kuramoto-Sivashinsky equation, Ginzburg-Landau equation, it can make significant influence on the system stability. Moreover, the effect of anti-diffusion term in some KdV-type equations is dicussed in [4].

State feedback stabilizing of a class of LKdVA is considered in [5], which can serve as the first part of a full result, with this paper as the second part. This paper is devoted to stabilizing the class of LKdVA by output feedback boundary control. Output feedback problems (see, e.g., [6], [7], [8], [9], [10]) are usually more applicable and/or more cost saving than state feedback problems. Moreover, since non-collocated control is generally preferable over collocated control in practice and performance, only non-collocated case is considered in this paper.

Difficulties in the LKdVA backstepping control design give rise to novel design and system analysis approaches. Firstly, it is not quite possible to use only a Volterra integral transformation (see, e.g., [11], [12], [13]) to convert this original system into an exponentially stable target system. In order to compensate for the presence of the anti-diffusion term, we apply another coordinate transformation [5] before using the Volterra integral transformation. Secondly and most important, because of the third order partial derivatives in the kernel function systems of the backstepping transformations, we need all the n-th, $n \in \mathbb{R}^+$, order partial derivatives for one of the kernel function successive approximation terms in each iteration of the successive approximation. The usual mathematical induction (with constant number of induction terms in the process) in successive approximation could not solve this problem. We employ a mathematical induction process in which the number of induction terms increases while the induction proceeds. Thirdly, it is not trivial to prove the invertibility of the kernel function for the transformation between the observer error system and the corresponding exponentially stable target system. This problem is resolved while solving state feedback control problem for a critical case of the LKdVA.

The remaining parts of this paper are organized as follows. In Section *II*, problem formulation is presented. In Section *III*, an observer is designed and the observer error systems are exponentially stable with an arbitrary decay rate. Output feedback control problem is considered in Section *IV*, where the closed-loop control systems are also proved to be exponentially stable with arbitrary decay rate. Some conclusion and possible future work are given in Section *V*. Existence, regularity and invertibility of the kernel function in Section *III* are shown from a preliminary state feedback boundary stabilizing problem in the Appendix.

II. PROBLEM FORMULATION

The problem we are concerned with is stabilizing the following class of LKdVA with boundary control and non-collocated observation:

$$u_t(x,t) = u_{xxx}(x,t) + \lambda_2 u_{xx}(x,t) + \lambda_1 u_x(x,t) + \lambda_0 u(x,t), \quad x \in (0,L) \tag{1}$$

$$u_x(0,t) = \lambda_3 u(0,t) \tag{2}$$

$$u_{xx}(0,t) = \lambda_4 u(0,t) \tag{3}$$

$$u(L,t) = U(t) \tag{4}$$

$$y(t) = u(0,t), \tag{5}$$

where $u(x,t) \in \mathbb{R}$ is the system state, $U(t)$ is the control input, $y(t)$ is the measured output, and $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ are known constants which can take any values. The control objective is to exponentially stabilize the system to zero in energy state space, and our control method is through observer-based boundary output feedback.

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III. Observer design

With the available measured output data (5), we first consider designing an observer to recover the full state of the system $(1)-(4)$.

The following observer is a "copy of the plant plus output injection terms":

$$\begin{aligned}\hat{u}_t(x,t)=&\hat{u}_{xxx}(x,t)+\lambda_2\hat{u}_{xx}(x,t)+\lambda_1\hat{u}_x(x,t)+\lambda_0\hat{u}(x,t)\\&-c_0(x)(u(0,t)-\hat{u}(0,t)),x\in(0,L) \quad (6)\\ \hat{u}_x(0,t)=&\lambda_3 u(0,t)-c_1(u(0,t)-\hat{u}(0,t)) \quad (7)\\ \hat{u}_{xx}(0,t)=&\lambda_4 u(0,t)-c_2(u(0,t)-\hat{u}(0,t)) \quad (8)\\ \hat{u}(L,t)=&U(t), \quad (9)\end{aligned}$$

where the function $c_0(x)$ and the constants c_1, c_2 are to be determined.

Denote

$$\tilde{u}(x,t)=u(x,t)-\hat{u}(x,t), \quad (10)$$

then the observer error system is as follows:

$$\begin{aligned}\tilde{u}_t(x,t)=&\tilde{u}_{xxx}(x,t)+\lambda_2\tilde{u}_{xx}(x,t)+\lambda_1\tilde{u}_x(x,t)+\lambda_0\tilde{u}(x,t)\\&+c_0(x)\tilde{u}(0,t),x\in(0,L) \quad (11)\\ \tilde{u}_x(0,t)=&c_1\tilde{u}(0,t) \quad (12)\\ \tilde{u}_{xx}(0,t)=&c_2\tilde{u}(0,t) \quad (13)\\ \tilde{u}(L,t)=&0. \quad (14)\end{aligned}$$

Let

$$\tilde{v}(x,t)=\tilde{u}(x,t)e^{\tilde{\varepsilon}x}, \quad (15)$$

where

$$\tilde{\varepsilon}\leqslant\lambda_2/3, \quad (16)$$

then

$$\begin{aligned}\tilde{v}_t(x,t)=&\tilde{v}_{xxx}(x,t)+\tilde{\mu}_2\tilde{v}_{xx}(x,t)+\tilde{\mu}_1\tilde{v}_x(x,t)+\tilde{\mu}_0\tilde{v}(x,t)\\&+c_0(x)e^{\tilde{\varepsilon}x}\tilde{v}(0,t),x\in(0,L) \quad (17)\\ \tilde{v}_x(0,t)=&\tilde{\mu}_3\tilde{v}(0,t) \quad (18)\\ \tilde{v}_{xx}(0,t)=&\tilde{\mu}_4\tilde{v}(0,t) \quad (19)\\ \tilde{v}(L,t)=&0, \quad (20)\end{aligned}$$

where

$$\begin{aligned}\tilde{\mu}_2=&\lambda_2-3\tilde{\varepsilon}\geqslant 0 \quad (21)\\ \tilde{\mu}_1=&3\tilde{\varepsilon}^2-2\lambda_2\tilde{\varepsilon}+\lambda_1 \quad (22)\\ \tilde{\mu}_0=&-(\tilde{\varepsilon}^3-\lambda_2\tilde{\varepsilon}^2+\lambda_1\tilde{\varepsilon}-\lambda_0) \quad (23)\\ \tilde{\mu}_3=&\tilde{\varepsilon}+c_1 \quad (24)\\ \tilde{\mu}_4=&\tilde{\varepsilon}^2+2\tilde{\varepsilon}c_1+c_2. \quad (25)\end{aligned}$$

We would like to find a backstepping transformation:

$$\tilde{v}(x,t)=\tilde{w}(x,t)-\int_0^x\tilde{\kappa}(x,y)\tilde{w}(y,t)dy, \quad (26)$$

where the kernel function $\tilde{\kappa}(x,y)\in\mathbb{R}$ is to be determined, to transform the system $(17)-(20)$ into the following exponentially stable target system:

$$\begin{aligned}\tilde{w}_t(x,t)=&\tilde{w}_{xxx}(x,t)+\tilde{\mu}_2\tilde{w}_{xx}(x,t)+\tilde{\mu}_1\tilde{w}_x(x,t)+\tilde{\nu}_0\tilde{w}(x,t),\\&x\in(0,L) \quad (27)\\ \tilde{w}_x(0,t)=&\tilde{\mu}_3\tilde{w}(0,t) \quad (28)\\ \tilde{w}_{xx}(0,t)=&\tilde{\nu}_4\tilde{w}(0,t) \quad (29)\\ \tilde{w}(L,t)=&0 \quad (30)\end{aligned}$$

where we firstly choose

$$\tilde{\nu}_0<\frac{1}{4L^2}\tilde{\mu}_2=\frac{1}{4L^2}(\lambda_2-3\tilde{\varepsilon}), \quad (31)$$

then for arbitrarily chosen c_1, choose c_2 such that

$$\begin{aligned}c_2\geqslant&-\frac{1}{3}L\tilde{\varepsilon}^3+\left(\frac{1}{3}L\lambda_2+1\right)\tilde{\varepsilon}^2+\left(2c_1-\frac{1}{3}L\lambda_1\right)\tilde{\varepsilon}\\&-\frac{1}{3}L\tilde{\nu}_0+\frac{1}{2}c_1^2-\lambda_2c_1+\frac{1}{3}L\lambda_0-\frac{1}{2}\lambda_1, \quad (32)\end{aligned}$$

lastly choose

$$\begin{aligned}\tilde{\nu}_4=&\tilde{\mu}_4+\frac{\tilde{\nu}_0-\tilde{\mu}_0}{3}L\\=&\frac{1}{3}L\tilde{\varepsilon}^3+\left(1-\frac{1}{3}L\lambda_2\right)\tilde{\varepsilon}^2+\left(\frac{1}{3}L\lambda_1+2c_1\right)\tilde{\varepsilon}\\&+\frac{1}{3}L\tilde{\nu}_0-\frac{1}{3}L\lambda_0+c_2, \quad (33)\end{aligned}$$

thus

$$\tilde{\mu}_1+2\tilde{\mu}_2\tilde{\mu}_3-\tilde{\mu}_3^2+2\tilde{\nu}_4\geqslant 0. \quad (34)$$

Remark 1: Exponential stability of system $(27)-(30)$ with $(21),(31),(34)$ is proved in Theorem 1 of [5], and the exponential decay rate estimation $\frac{1}{4L^2}\tilde{\mu}_2-\tilde{\nu}_0$ can be arbitrarily large by choosing $\tilde{\nu}_0$ small/negative enough.

By calculation and comparison between the systems $(17)-(20)$ and $(27)-(30)$, the functions $\tilde{\kappa}(x,y)$ and $c_0(x)$ need to satisfy

$$\begin{aligned}&\tilde{\kappa}_{xxx}(x,y)+\tilde{\kappa}_{yyy}(x,y)+\tilde{\mu}_2(\tilde{\kappa}_{xx}(x,y)-\tilde{\kappa}_{yy}(x,y))\\&\quad+\tilde{\mu}_1(\tilde{\kappa}_x(x,y)+\tilde{\kappa}_y(x,y))=(\tilde{\nu}_0-\tilde{\mu}_0)\tilde{\kappa}(x,y) \quad (35)\\&\tilde{\kappa}(x,x)=0 \quad (36)\\&\tilde{\kappa}_x(x,x)=\frac{\tilde{\mu}_0-\tilde{\nu}_0}{3}(x-L) \quad (37)\\&\tilde{\kappa}(L,y)=0 \quad (38)\end{aligned}$$

and

$$\begin{aligned}c_0(x)=&\big[\tilde{\kappa}_{yy}(x,0)-(\tilde{\mu}_2+\tilde{\mu}_3)\tilde{\kappa}_y(x,0)\\&+(\tilde{\mu}_1+\tilde{\mu}_2\tilde{\mu}_3+\tilde{\nu}_4)\tilde{\kappa}(x,0)\big]e^{-\tilde{\varepsilon}x}. \quad (39)\end{aligned}$$

Let

$$\tilde{\kappa}(x,y)=\bar{\kappa}(\bar{x},\bar{y}), \quad (40)$$

where

$$\bar{x}=L-y,\ \bar{y}=L-x, \quad (41)$$

then

$$c_0(x) = \left[\bar{\kappa}_{\bar{x}\bar{x}}(L,L-x) + (\tilde{\mu}_2+\tilde{\mu}_3)\bar{\kappa}_{\bar{x}}(L,L-x) + (\tilde{\mu}_1+\tilde{\mu}_2\tilde{\mu}_3+\tilde{\nu}_4)\bar{\kappa}(L,L-x)\right]e^{-\tilde{\varepsilon}x}, \quad (42)$$

where $\bar{\kappa}(\bar{x},\bar{y})$ satisfies

$$\bar{\kappa}_{\bar{x}\bar{x}\bar{x}}(\bar{x},\bar{y}) + \bar{\kappa}_{\bar{y}\bar{y}\bar{y}}(\bar{x},\bar{y}) - \tilde{\mu}_2(\bar{\kappa}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{\kappa}_{\bar{y}\bar{y}}(\bar{x},\bar{y})) + \tilde{\mu}_1(\bar{\kappa}_{\bar{x}}(\bar{x},\bar{y}) + \bar{\kappa}_{\bar{y}}(\bar{x},\bar{y})) = (\tilde{\mu}_0 - \tilde{\nu}_0)\bar{\kappa}(\bar{x},\bar{y}) \quad (43)$$

$$\bar{\kappa}(\bar{x},\bar{x}) = 0 \quad (44)$$

$$\bar{\kappa}_{\bar{x}}(\bar{x},\bar{x}) = \frac{\tilde{\nu}_0 - \tilde{\mu}_0}{3}\bar{x} \quad (45)$$

$$\bar{\kappa}(\bar{x},0) = 0. \quad (46)$$

The above PDE is in class of $(108)-(111)$ from the Appendix (with $\tilde{\mu}_0,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\nu}_0$ replaced by $\mu_0,\mu_1,-\mu_2,\nu_0$ respectively). Thus, existence, regularity and invertibility of $\bar{\kappa}(\bar{x},\bar{y})$ and also $\tilde{\kappa}(x,y)$ follow similarly as in Appendix.

Since the transformation (15) is also invertible, the following main theorem can be obtained.

Theorem 1: For any initial data $u(\cdot,0),\hat{u}(\cdot,0) \in L^2(0,L)$, the observer $(6)-(9)$, with constant c_1 arbitrarily chosen firstly, then constant c_2 chosen from $(32),(16),(31)$ and lastly function $c_0(x)$ derived by $(42)-(46),(21)-(24),(33)$, guarantees that the observer error system $(11)-(14)$ has a unique (mild) solution

$$\tilde{u}(\cdot,t) \in C([0,\infty);L^2(0,L)), \quad (47)$$

and there exist positive constants $\tilde{M},\tilde{\rho}$ such that

$$\|\tilde{u}(\cdot,t)\|_{L^2(0,L)} \leqslant \tilde{M}e^{-\tilde{\rho}t}\|\tilde{u}(\cdot,0)\|_{L^2(0,L)}. \quad (48)$$

Moreover, if $\tilde{u}(\cdot,0)$ satisfies boundary compatibility condition, then

$$\tilde{u}(\cdot,t) \in C^1([0,\infty);L^2(0,L)) \quad (49)$$

is the classical solution to $(11)-(14)$.

IV. OUTPUT FEEDBACK STABILIZATION

Based on the full state data recovered by the observer $(6)-(9)$, we now deal with the output feedback stabilization problem of the system $(1)-(5)$. Consider the observer $(6)-(9)$ with

$$U(t) = \int_0^L \kappa(L,y)\hat{u}(y,t)e^{\varepsilon(y-L)}dy, \quad (50)$$

where the constant ε and function $\kappa(x,y)$ are from [5], that is,

$$\varepsilon \leqslant \lambda_2/3, \quad (51)$$

and $\kappa(x,y)$ satisfies

$$\kappa_{xxx}(x,y) + \kappa_{yyy}(x,y) + \mu_2(\kappa_{xx}(x,y) - \kappa_{yy}(x,y)) + \mu_1(\kappa_x(x,y) + \kappa_y(x,y)) = (\mu_0 - \nu_0)\kappa(x,y) \quad (52)$$

$$\kappa(x,x) = \mu_3 - \nu_3 \quad (53)$$

$$\kappa_x(x,x) = \frac{\nu_0 - \mu_0}{3}x - (\mu_3-\nu_3)\mu_3 + \mu_4 - \nu_4 \quad (54)$$

$$\kappa_{yy}(x,0) - (\mu_2+\mu_3)\kappa_y(x,0) + (\mu_1+\mu_2\mu_3+\mu_4)\kappa(x,0) = 0, \quad (55)$$

where

$$\mu_0 = -\varepsilon^3 + \lambda_2\varepsilon^2 - \lambda_1\varepsilon + \lambda_0 \quad (56)$$

$$\mu_1 = 3\varepsilon^2 - 2\lambda_2\varepsilon + \lambda_1 \quad (57)$$

$$\mu_2 = -3\varepsilon + \lambda_2 \quad (58)$$

$$\mu_3 = \varepsilon + \lambda_3 \quad (59)$$

$$\mu_4 = \varepsilon^2 + 2\lambda_3\varepsilon + \lambda_4 \quad (60)$$

and ν_0,ν_3,ν_4 are chosen such that

$$\nu_0 < \frac{1}{4L^2}\mu_2, \ \mu_1 + 2\mu_2\nu_3 - \nu_3^2 + 2\nu_4 \geqslant 0. \quad (61)$$

Let

$$\hat{v}(x,t) = \hat{u}(x,t)e^{\varepsilon x}, \quad (62)$$

then

$$\hat{v}_t(x,t) = \hat{v}_{xxx}(x,t) + \mu_2\hat{v}_{xx}(x,t) + \mu_1\hat{v}_x(x,t) + \mu_0\hat{v}(x,t) - c_0(x)e^{\varepsilon x}\tilde{v}(0,t), x \in (0,L) \quad (63)$$

$$\hat{v}_x(0,t) = \mu_3\hat{v}(0,t) + \mu_5\tilde{v}(0,t) \quad (64)$$

$$\hat{v}_{xx}(0,t) = \mu_4\hat{v}(0,t) + \mu_6\tilde{v}(0,t) \quad (65)$$

$$\hat{v}(L,t) = \int_0^L \kappa(L,y)\hat{v}(y,t)dy, \quad (66)$$

where

$$\mu_5 = \lambda_3 - c_1 \quad (67)$$

$$\mu_6 = 2\varepsilon(\lambda_3 - c_1) + \lambda_4 - c_2. \quad (68)$$

Apply the invertible transformation $\hat{v} \mapsto \hat{w}$:

$$\hat{w}(x,t) = \hat{v}(x,t) - \int_0^x \kappa(x,y)\hat{v}(y,t)dy, \quad (69)$$

then we get the following class of systems:

$$\hat{w}_t(x,t) = \hat{w}_{xxx}(x,t) + \mu_2\hat{w}_{xx}(x,t) + \mu_1\hat{w}_x(x,t) + \nu_0\hat{w}(x,t) - \left[\mu_5\kappa_y(x,0) - (\mu_2\mu_5+\mu_6)\kappa(x,0) + c_0(x)e^{\varepsilon x} - \int_0^x \kappa(x,y)c_0(y)e^{\varepsilon y}dy\right]\tilde{w}(0,t), x \in (0,L) \quad (70)$$

$$\hat{w}_x(0,t) = \nu_3\hat{w}(0,t) + \nu_5\tilde{w}(0,t) \quad (71)$$

$$\hat{w}_{xx}(0,t) = \nu_4\hat{w}(0,t) + \nu_6\tilde{w}(0,t) \quad (72)$$

$$\hat{w}(L,t) = 0, \quad (73)$$

where

$$\nu_5 = \mu_5 \quad (74)$$

$$\nu_6 = -(\mu_3 - \nu_3)\mu_5 + \mu_6. \quad (75)$$

The $\tilde{w}$-system $(27)-(30)$ and homogenous part of the $\hat{w}$-system $(70)-(73)$ are both exponentially stable. The interconnection of two systems $(\hat{w},\tilde{w})$ is a cascade, and the combined system $(70)-(73),(27)-(30)$ is exponentially stable, which is to be proved.

Consider the Hilbert space

$$\mathbb{H} = L^2(0,L) \times L^2(0,L) \quad (76)$$

with an inner product

$$< (f_1,g_1),(f_2,g_2) >= \int_0^L a f_1(x)\overline{f_2(x)} + g_1(x)\overline{g_2(x)}dx, \quad \forall (f_1,g_1),(f_2,g_2) \in \mathbb{H}, \tag{77}$$

where $a > 0$ is a constant to be determined.

Define the system operator $\mathbf{A} : D(\mathbf{A})(\subset \mathbb{H}) \to \mathbb{H}$ as follows:

$$\begin{aligned}\mathbf{A}(f,g) = \Big(& f''' + \mu_2 f'' + \mu_1 f' + \nu_0 f \\ & - \Big[\mu_5 \kappa_y(x,0) - (\mu_2\mu_5 + \mu_6)\kappa(x,0) + c_0(x)e^{\varepsilon x} \\ & - \int_0^x \kappa(x,y)c_0(y)e^{\varepsilon y}dy\Big] g(0), \\ & g''' + \tilde{\mu}_2 g'' + \tilde{\mu}_1 g' + \tilde{\nu}_0 g\Big), \forall (f,g) \in D(\mathbf{A}),\end{aligned} \tag{78}$$

$$\begin{aligned}D(\mathbf{A}) = \{& (f,g) \in H^3(0,L) \times H^3(0,L) | f(L) = g(L) = 0, \\ & f'(0) = \nu_3 f(0) + \nu_5 g(0), g'(0) = \tilde{\mu}_3 g(0), \\ & f''(0) = \nu_4 f(0) + \nu_6 g(0), g''(0) = \tilde{\nu}_4 g(0)\},\end{aligned} \tag{79}$$

then the $(\hat{w},\tilde{w})$-system $(70)-(73),(27)-(30)$ can be written as an evolution equation in $\mathbb{H}$:

$$\frac{d(\hat{w}(\cdot,t),\tilde{w}(\cdot,t))}{dt} = \mathbf{A}(\hat{w}(\cdot,t),\tilde{w}(\cdot,t)). \tag{80}$$

Lemma 1: If

$$\begin{pmatrix} 1 & \nu_3 & \nu_4 \end{pmatrix} e^{D_1 L} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0,$$

$$\begin{pmatrix} 1 & \tilde{\mu}_3 & \tilde{\nu}_4 \end{pmatrix} e^{D_2 L} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0, \tag{81}$$

where

$$D_1 = \begin{pmatrix} 0 & 0 & -\nu_0 \\ 1 & 0 & -\mu_1 \\ 0 & 1 & -\mu_2 \end{pmatrix},\ D_2 = \begin{pmatrix} 0 & 0 & -\tilde{\nu}_0 \\ 1 & 0 & -\tilde{\mu}_1 \\ 0 & 1 & -\tilde{\mu}_2 \end{pmatrix}, \tag{82}$$

then $\mathbf{A}^{-1}$ exists and is compact on $\mathbb{H}$. Hence, $\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$, consists of isolated eigenvalues only.

Let $(f,g) \in D(\mathbf{A})$, then

$$\begin{aligned}& Re <\mathbf{A}(f,g),(f,g)> \\ \leqslant & a(\nu_0 - \frac{1}{4L^2}\mu_2 + \frac{\theta^2 L}{4m_1})\|f\|^2_{L^2(0,L)} \\ & + (\tilde{\nu}_0 - \frac{1}{4L^2}\tilde{\mu}_2)\|g\|^2_{L^2(0,L)} \\ & - a\big(\nu_4 + \mu_2\nu_3 - \nu_3^2 + \frac{1}{2}\mu_1 - (\nu_6 + \mu_2\nu_5)^2 m_2\big)|f(0)|^2 \\ & - (\tilde{\nu}_4 + \tilde{\mu}_2\tilde{\mu}_3 - \frac{1}{2}\tilde{\mu}_3^2 + \frac{1}{2}\tilde{\mu}_1 - a(\nu_5^2 + m_1 + \frac{1}{4m_2}))|g(0)|^2\end{aligned} \tag{83}$$

where

$$\begin{aligned}\theta = \sup_{0\leqslant x\leqslant L} \Big| & (\mu_5 \kappa_y(x,0) - (\mu_2\mu_5 + \mu_6)\kappa(x,0) + c_0(x)e^{\varepsilon x} \\ & - \int_0^x \kappa(x,y)c_0(y)e^{\varepsilon y}dy \Big|\end{aligned} \tag{84}$$

and the constants $m_1 > 0, m_2 > 0$ are to be determined.

First choose

$$m_1 > \frac{\theta^2 L}{\mu_2/4L^2 - \nu_0} \tag{85}$$

$$0 < m_2 \leqslant \frac{\nu_4 + \mu_2\nu_3 - \nu_3^2 + \frac{1}{2}\mu_1}{(\nu_6 + \mu_2\nu_5)^2}, \tag{86}$$

then choose

$$0 < a \leqslant \frac{\tilde{\nu}_4 + \tilde{\mu}_2\tilde{\mu}_3 - \frac{1}{2}\tilde{\mu}_3^2 + \frac{1}{2}\tilde{\mu}_1}{\nu_5^2 + m_1 + \frac{1}{4m_2}}, \tag{87}$$

we get

$$Re <\mathbf{A}(f,g),(f,g)> \leqslant -\rho\|(f,g)\|^2_{\mathbb{H}},\ \forall (f,g) \in D(\mathbf{A}), \tag{88}$$

where

$$\rho = \min\left\{\frac{1}{4L^2}\mu_2 - \nu_0 - \frac{\theta^2 L}{4m_1},\ \frac{1}{4L^2}\tilde{\mu}_2 - \tilde{\nu}_0\right\} > 0. \tag{89}$$

We will drop the subscripts from the norms being used in the sequel when clear from the context.

Lemma 2: $\mathbf{A}$ is dissipative in $\mathbb{H}$, and $\mathbf{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$ of contractions in $\mathbb{H}$. For each $\lambda \in \sigma(\mathbf{A})$, $Re\lambda < 0$.

Define a Lyapunov function

$$V_w(t) = \frac{a}{2}\|\hat{w}(\cdot,t)\|^2 + \frac{1}{2}\|\tilde{w}(\cdot,t)\|^2, \tag{90}$$

then we can get

$$\dot{V}_w(t) \leqslant -2\rho V_w(t). \tag{91}$$

Since $\mathbf{A}$ generates a C_0-semigroup $e^{\mathbf{A}t}$, this semigroup must be exponentially stable.

Theorem 2: $\mathbf{A}$ generates an exponentially stable C_0 semigroup on $\mathbb{H}$. For any initial data $(\hat{w}(\cdot,0),\tilde{w}(\cdot,0)) \in \mathbb{H}$, there exists a unique (mild) solution to the transformed $(\hat{w},\tilde{w})$-system $(70)-(73)$, $(27)-(30)$ such that

$$(\hat{w}(\cdot,t),\tilde{w}(\cdot,t)) \in C([0,\infty);\mathbb{H}), \tag{92}$$

and

$$\|(\hat{w}(\cdot,t),\tilde{w}(\cdot,t))\| \leqslant e^{-\rho t}\|(\hat{w}(\cdot,0),\tilde{w}(\cdot,0))\|. \tag{93}$$

Moreover, if $(\hat{w}(\cdot,0),\tilde{w}(\cdot,0)) \in D(\mathbf{A})$, then

$$(\hat{w}(\cdot,t),\tilde{w}(\cdot,t)) \in C^1([0,\infty);\mathbb{H}) \tag{94}$$

is the classical solution.

From invertibility of the transforms $(15),(26),(62),(69)$, this following main theorem holds.

Theorem 3: For any initial data $(u(\cdot,0),\hat{u}(\cdot,0)) \in \mathbb{H}$, there exists a unique (mild) solution to the closed-loop $(u,\hat{u})$-system $(1)-(5),(6)-(9)$ (in which constant c_1 is arbitrarily chosen firstly, then constant c_2 is chosen from $(32),(16),(31)$ and lastly function $c_0(x)$ is derived by $(42)-(46),(21)-(24),(33)$) with the controller determined from $(50)-(61)$, such that

$$(u(\cdot,t),\hat{u}(\cdot,t)) \in C([0,\infty);\mathbb{H}), \tag{95}$$

and there exists a positive constant M_u such that

$$\|(u(\cdot,t),\hat{u}(\cdot,t))\| \leqslant M_u e^{-\rho t}\|(u(\cdot,0),\hat{u}(\cdot,0))\|. \tag{96}$$

Moreover, if $(u(\cdot,0),\hat{u}(\cdot,0))$ satisfies the boundary compatibility condition, then

$$(u(\cdot,t),\hat{u}(\cdot,t)) \in C^1([0,\infty);\mathbb{H}) \tag{97}$$

is the classical solution.

Remark 2: ρ is an exponential decay rate estimate, which can be arbitrarily large by choosing $\nu_0, \tilde{\nu}_0$ small enough and then choosing m_1 large enough.

V. Conclusion and Future Work

A control design for stabilizing a class of LKdVA is presented, by means of the non-collocated boundary feedback. An arbitrary exponential decay rate of the resulting closed-loop control systems is achieved.

For future work, subclasses of LKdVA which are Riesz spectral systems are to be considered, for which more useful results might be derived. Also, stabilization of the linearized KdV systems with spatially varying coefficients can be an interesting problem. Another problem is stabilization for the case when some coefficients of the systems are unknown.

Appendix

Consider the following subclass of LKdVA

$$\begin{aligned} v_t(x,t) =& v_{xxx}(x,t)+\mu_2 v_{xx}(x,t)+\mu_1 v_x(x,t)+\mu_0 v(x,t), \\ & x\in(0,L) && (98)\\ v(0,t) =& 0 && (99)\\ v_x(0,t) =& 0 && (100)\\ v(L,t) =& V(t) && (101) \end{aligned}$$

and an exponentially stable target system

$$\begin{aligned} w_t(x,t) =& w_{xxx}(x,t)+\mu_2 w_{xx}(x,t)+\mu_1 w_x(x,t)+\nu_0 w(x,t), \\ & x\in(0,L) && (102)\\ w(0,t) =& 0 && (103)\\ w_x(0,t) =& 0 && (104)\\ w(L,t) =& 0, && (105) \end{aligned}$$

which can be derived from [5] by choosing $\nu_1=\mu_1, \nu_2=\mu_2, \nu_3=\mu_3$ and taking a critical case of $\nu_4=\mu_4=+\infty$. Here μ_0,μ_1 are arbitrary known constants and

$$\mu_2 \geqslant 0, \nu_0 < \frac{1}{4L^2}\mu_2. \tag{106}$$

In order to obtain the state feedback controller, we use the proposed transformation $v\mapsto w$:

$$w(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)dy \tag{107}$$

with the kernel function $k(x,y)\in\mathbb{R}$ to satisfy:

$$\begin{aligned} & k_{xxx}(x,y)+k_{yyy}(x,y)+\mu_2(k_{xx}(x,y)-k_{yy}(x,y)) \\ & \quad +\mu_1(k_x(x,y)+k_y(x,y)) = (\mu_0-\nu_0)k(x,y) && (108)\\ & k(x,x)=0 && (109)\\ & k_x(x,x)=\frac{\nu_0-\mu_0}{3}x && (110)\\ & k(x,0)=0, && (111) \end{aligned}$$

which is the corresponding special case of $(57)-(60)$ in [5]. Thus, the following lemma holds:

Lemma 3: The system of equations $(108)-(111)$ has a unique C^3 solution, and the system of equations for kernel function $l(x,y)$ of the inverse transformation $w\mapsto v$:

$$v(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy, \tag{112}$$

also has a unique C^3 solution.

From continuity of the transformations $(107),(112)$ and exponential stability of the system $(102)-(105)$, the following lemma holds.

Theorem 4: For any initial data $v(\cdot,0)\in L^2(0,L)$, there exists a unique (mild) solution to the closed-loop system $(98)-(101)$ (where μ_0, μ_1 are arbitrarily chosen, and μ_2, ν_0 are chosen from (106)) with the controller

$$V(t) = \int_0^L k(L,y)v(y,t)dy \tag{113}$$

such that

$$v(\cdot,t)\in C([0,\infty);L^2(0,L)), \tag{114}$$

and there exist positive constants M_v,ρ_v such that

$$\|v(\cdot,t)\| \leqslant M_v e^{-\rho_v t}\|v(\cdot,0)\|. \tag{115}$$

Moreover, if $v(\cdot,0)$ satisfies the boundary compatibility condition, then

$$v(\cdot,t)\in C^1([0,\infty);L^2(0,L)) \tag{116}$$

is the classical solution.

References

[1] L. Rosier, "Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 2, pp. 33–55, 1997.

[2] D. L. Russell and B.-Y. Zhang, "Exact controllability and stabilizability of the Korteweg-de Vries equation," *Trans. Amer. Math. Soc.*, vol. 348, pp. 3643–3672, 1996.

[3] E. Cerpa and J. Coron, "Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition," *IEEE Transactions on Automatic Control*, to appear.

[4] D. M. Ambrose and J. D. Wright, "Dispersion vs. anti-diffusion: well-posedness in variable coefficient and quasilinear equations of KdV-type," *Indiana University Mathematics Journal*, vol. 62, no. 4, pp. 1237–1281, 2013.

[5] S. Tang and M. Krstic, "Stabilization of Linearized Korteweg-de Vries systems with Anti-diffusion," *Proceedings of the 2013 American Control Conference*, pp. 3302 – 3307, 2013.

[6] M. Krstic, B. Guo, A. Balogh, and A. Smyshlyaev, "Control of a tip-force destabilized shear beam by observer-based boundary feedback," *SIAM Journal on Control and Optimization*, vol. 47, no. 2, pp. 553–574, 2008.

[7] ——, "Output-feedback stabilization of an unstable wave equation," *Automatica*, vol. 44, pp. 63–74, 2008.

[8] A. Smyshlyaev and M. Krstic, "Explicit state and output feedback boundary controllers for partial differential equations," *Journal of Automatic Control*, vol. 13, no. 2, pp. 1–9, 2003.

[9] ——, "Backstepping observers for a class of parabolic pdes," *Systems & Control Letters*, vol. 54, pp. 613–625, 2005.

[10] S. Tang and C. Xie, "State and output feedback boundary control for a coupled PDE-ODE system," *Systems & Control Letters*, vol. 60, pp. 540–545, 2011.

[11] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*. SIAM, 2008.

[12] ——, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Systems & Control Letters*, vol. 57, pp. 750–758, 2008.

[13] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhauser, 2009.