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Stabilization of Linearized Korteweg-de Vries Systems with Anti-diffusion

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Abstract— **In this paper, backstepping boundary controllers are designed for a class of linearized Korteweg-de Vries systems with possible anti-diffusion, and the resulting closed-loop systems can achieve arbitrary exponential decay rate. Semigroup of linear operators is constructed in analyzing well-posedness and stability of the target systems, and mathematical induction is used in proving existence of kernel functions. An example is also presented, which illustrates performance of the controller. The decay rate estimate derived in this paper is not necessarily equal to decay rate, which can be seen from the appendix.**

Index Terms— **Linearized Korteweg-de Vries systems; Anti-diffusion; Backstepping; Arbitrary exponential decay rate; Semigroup of linear operators.**

I. Introduction

Korteweg-de Vries equation (KdV equation for short) is a nonlinear partial differential equation (PDE for short) of third order, which can be used to model waves on shallow water surfaces and ion-acoustic waves in plasmas. Controllability and stabilization of KdV equations are topics of active research (see, e.g., [1], [2], [3]). This paper is devoted to stabilizing a class of linearized KdV systems with possible anti-diffusion by backstepping boundary control.

The method of backstepping can be used for stabilizing unstable PDE systems. For example, in [4], [5], [6], backstepping boundary controllers are designed for some unstable parabolic, hyperbolic and even complex-valued PDEs, etc, and the resulting closed-loop control systems are exponentially stable.

Arbitrary exponential decay rate is desirable in engineering, which has also obtained much attention from scientists (see, e. g. [7], [8], [9], [3]). One elegant method to analyze stability of PDE systems is through applying theory of semigroups of linear operators (see, e. g, [10], [11]).

This paper is organized as follows. In Section II, problem formulation is presented. Well-posedness and exponential stability with arbitrary decay rate of a class of target systems are analyzed and proved in Section III, where theory of semigroups of linear operators is applied. In Section IV, existence of kernel functions for backstepping boundary controllers is proved by mathematical induction, and direct and inverse transformation between the v-system and w-system are derived. Then, exponential stability with arbitrary decay rate of the resulting closed-loop control systems is proved. Moreover, an example is presented in Section V. Some conclusion and possible future work are given in Section VI. Exponential decay rate estimate derived in this paper is not necessarily equal to decay rate, as can be seen from Appendix .

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II. Problem formulation

Consider the following class of linearized KdV control systems with anti-diffusion

$$
\begin{aligned}
u_t(x,t) =& u_{xxx}(x,t) + \lambda_2 u_{xx}(x,t) + \lambda_1 u_x(x,t) \\
&+ \lambda_0 u(x,t), x \in (0,L) \qquad (1)\\
u_x(0,t) =& \lambda_3 u(0,t) \qquad (2)\\
u_{xx}(0,t) =& \lambda_4 u(0,t) \qquad (3)\\
u(L,t) =& U(t). \qquad (4)
\end{aligned}
$$

Remark 1: $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ can take any values. We call this class of systems "with anti-diffusion" only to emphasize that λ_2 is allowed to be negative.

Let

$$v(x,t) = u(x,t)e^{\varepsilon x}, \qquad (5)$$

where ε is to be determined later, then we get the following systems

$$
\begin{aligned}
v_t(x,t) =& v_{xxx}(x,t) + \mu_2 v_{xx}(x,t) + \mu_1 v_x(x,t) \\
&+ \mu_0 v(x,t), x \in (0,L) \qquad (6)\\
v_x(0,t) =& \mu_3 v(0,t) \qquad (7)\\
v_{xx}(0,t) =& \mu_4 v(0,t) \qquad (8)\\
v(L,t) =& V(t), \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_0 =& -\varepsilon^3 + \lambda_2 \varepsilon^2 - \lambda_1 \varepsilon + \lambda_0 \qquad (10)\\
\mu_1 =& 3\varepsilon^2 - 2\lambda_2 \varepsilon + \lambda_1 \qquad (11)\\
\mu_2 =& -3\varepsilon + \lambda_2 \qquad (12)\\
\mu_3 =& \varepsilon + \lambda_3 \qquad (13)\\
\mu_4 =& \varepsilon^2 + 2\lambda_3 \varepsilon + \lambda_4 \qquad (14)\\
V(t) =& U(t)e^{\varepsilon L}. \qquad (15)
\end{aligned}
$$

III. Target system

Consider the following class of target systems

$$
\begin{aligned}
w_t(x,t) =& w_{xxx}(x,t) + \nu_2 w_{xx}(x,t) + \nu_1 w_x(x,t) \\
&+ \nu_0 w(x,t), x \in (0,L) \qquad (16)\\
w_x(0,t) =& \nu_3 w(0,t) \qquad (17)\\
w_{xx}(0,t) =& \nu_4 w(0,t) \qquad (18)\\
w(L,t) =& 0, \qquad (19)
\end{aligned}
$$

where

$$\nu_2 \geqslant 0,\ \nu_1+2\nu_2\nu_3-\nu_3^2+2\nu_4 \geqslant 0,\ \nu_0 < \frac{1}{4L^2}\nu_2. \quad (20)$$

Remark 2: The system of inequalities (20) is a sufficient but not necessary condition for the target systems (16) − (19) to be exponentially stable. Moreover, from what will be stated later in Section IV, we will choose

$$\nu_1 = \mu_1, \nu_2 = \mu_2. \quad (21)$$

There are two ways of making design choices for the parameters in order to satisfy (20) and (21):

(1). First choose a $\varepsilon \leqslant \lambda_2/3$ such that $\mu_2 \geqslant 0$. Then, choose ν_0, ν_3, ν_4 such that

$$\nu_1+2\nu_2\nu_3-\nu_3^2+2\nu_4 \geqslant 0,\ \nu_0 < \frac{1}{4L^2}\nu_2, \quad (22)$$

where ν_1 and ν_2 are known.

(2). First choose any ν_3, ν_4, then choose ε from the following system of inequalities:

$$-3\varepsilon+\lambda_2 \geqslant 0 \quad (23)$$

$$3\varepsilon^2-2(\lambda_2+3\nu_3)\varepsilon+\lambda_1+2\lambda_2\nu_3-\nu_3^2+2\nu_4 \geqslant 0, \quad (24)$$

which always has solutions. Thus, ν_1, ν_2 are known. Last, choose

$$\nu_0 < \frac{1}{4L^2}\nu_2. \quad (25)$$

To establish stability and well-posedness for this class of systems, consider the state Hilbert space $\mathbf{H} = L^2(0,L)$. Define the system operator $\mathscr{A} : D(\mathscr{A})(\subset \mathbf{H}) \to \mathbf{H}$ as follows:

$$\mathscr{A} f = f''' + \nu_2 f'' + \nu_1 f' + \nu_0 f, \forall f \in D(\mathscr{A}), \quad (26)$$

$$D(\mathscr{A}) = \{f \in H^3(0,L) \mid f'(0) = \nu_3 f(0), \quad f''(0) = \nu_4 f(0), f(L) = 0\}, \quad (27)$$

then the system (16) − (19) can be written as an evolution equation in $\mathbf{H}$:

$$\frac{dw(\cdot,t)}{dt} = \mathscr{A} w(\cdot,t). \quad (28)$$

Lemma 1: If $\begin{pmatrix} 1 & \nu_3 & \nu_4 \end{pmatrix} e^{DL} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is nonzero, where

$$D = \begin{pmatrix} 0 & 0 & -\nu_0 \\ 1 & 0 & -\nu_1 \\ 0 & 1 & -\nu_2 \end{pmatrix}, \quad (29)$$

then $\mathscr{A}^{-1}$ exists and is compact on $\mathbf{H}$. Hence, $\sigma(\mathscr{A})$, the spectrum of $\mathscr{A}$, consists of isolated eigenvalues only: $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A})$, where $\sigma_p(\mathscr{A})$ denotes the set of eigenvalues of $\mathscr{A}$. Moreover, each $\lambda \in \sigma(\mathscr{A})$ is geometrically simple and satisfies the characteristic equation

$$\begin{aligned} 0 = & e^{\sigma_1 L}(\sigma_2-\sigma_3)(\sigma_2\sigma_3-\nu_3(\sigma_2+\sigma_3)+\nu_4) \\ & + e^{\sigma_2 L}(\sigma_3-\sigma_1)(\sigma_3\sigma_1-\nu_3(\sigma_3+\sigma_1)+\nu_4) \\ & + e^{\sigma_3 L}(\sigma_1-\sigma_2)(\sigma_1\sigma_2-\nu_3(\sigma_1+\sigma_2)+\nu_4), \end{aligned} \quad (30)$$

where

$$\sigma_1 = -\frac{\nu_2}{3} + \alpha + \beta \quad (31)$$

$$\sigma_2 = -\frac{\nu_2}{3} + \omega\alpha + \omega^2\beta \ (with\ \omega = e^{2/3\pi i}) \quad (32)$$

$$\sigma_3 = -\frac{\nu_2}{3} + \omega^2\alpha + \omega\beta, \quad (33)$$

and

$$\alpha = \sqrt[3]{\tau_1+\sqrt{\tau_1^2+\tau_2^3}},\ \beta = \sqrt[3]{\tau_1-\sqrt{\tau_1^2+\tau_2^3}}, \quad (34)$$

$$\tau_1 = \frac{\nu_1\nu_2}{6} - \frac{\nu_2^3}{27} - \frac{\nu_0-\lambda}{2},\ \tau_2 = \frac{\nu_1}{3} - \frac{\nu_2^2}{9}. \quad (35)$$

An eigenfunction f corresponding to λ is

$$\begin{aligned} f(x) = & (\sigma_2-\sigma_3)(\sigma_2\sigma_3-\nu_3(\sigma_2+\sigma_3)+\nu_4)e^{\sigma_1 x} \\ & + (\sigma_3-\sigma_1)(\sigma_3\sigma_1-\nu_3(\sigma_3+\sigma_1)+\nu_4)e^{\sigma_2 x} \\ & + (\sigma_1-\sigma_2)(\sigma_1\sigma_2-\nu_3(\sigma_1+\sigma_2)+\nu_4)e^{\sigma_3 x}. \end{aligned} \quad (36)$$

Proof: (Part 1) By calculation, we get

$$\mathscr{A}^{-1} f = f_1,\ \forall f \in \mathbf{H}, \quad (37)$$

$$\begin{aligned} f_1(x) = & f_1(0)\begin{pmatrix} 1 & \nu_3 & \nu_4 \end{pmatrix} e^{Dx} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ & + \int_0^x f(\tau)\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} e^{D(x-\tau)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} d\tau, \end{aligned} \quad (38)$$

where

$$\begin{aligned} f_1(0) = & -\int_0^L f(\tau)\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} e^{D(L-\tau)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} d\tau \\ & \times \left(\begin{pmatrix} 1 & \nu_3 & \nu_4 \end{pmatrix} e^{DL} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)^{-1}. \end{aligned} \quad (39)$$

Hence we get the unique $f_1 \in D(\mathscr{A})$ and thus $\mathscr{A}^{-1}$ exists and is compact on $\mathbf{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathscr{A})$, the spectrum of $\mathscr{A}$, consists of isolated eigenvalues only.

(Part 2) For any $\lambda \in \sigma_p(\mathscr{A})$, we have

$$\mathscr{A} f = f''' + \nu_2 f'' + \nu_1 f' + \nu_0 f = \lambda f \quad (40)$$

$$f'(0) = \nu_3 f(0), f''(0) = \nu_4 f(0), f(L) = 0, \quad (41)$$

which has at least one nonzero solution. If it has two linearly independent solutions f_1, f_2, then there exists constants a, b ($a^2+b^2 \neq 0$) such that $af_1(0)+bf_2(0) = 0$. Thus, $f = af_1 + bf_2$ satisfies

$$\mathscr{A} f = f''' + \nu_2 f'' + \nu_1 f' + \nu_0 f = \lambda f \quad (42)$$

$$f(0) = f'(0) = f''(0) = f(L) = 0, \quad (43)$$

which has only zero solution. Hence, $af_1 + bf_2 \equiv 0$, which contradicts with the assumption. Therefore, each $\lambda \in \sigma_p(\mathscr{A})$ is geometrically simple.

(Part 3) For any $\lambda \in \sigma_p(\mathscr{A})$, from (40), we have

$$f(x) = c_1 e^{\sigma_1 x} + c_2 e^{\sigma_2 x} + c_3 e^{\sigma_3 x}\ (c_1^2+c_2^2+c_3^2 \neq 0). \quad (44)$$

From (41), we get

$$\begin{vmatrix} \sigma_1-\nu_3 & \sigma_2-\nu_3 & \sigma_3-\nu_3 \\ \sigma_1^2-\nu_4 & \sigma_2^2-\nu_4 & \sigma_3^2-\nu_4 \\ e^{\sigma_1 L} & e^{\sigma_2 L} & e^{\sigma_3 L} \end{vmatrix} = 0 \tag{45}$$

and the characteristic equation is (30). We can also derive the corresponding eigenfunction (36). ■

Lemma 2: $\mathscr{A}$ is dissipative in $\mathbf{H}$, and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathbf{H}$.

Proof: Let $f \in D(\mathscr{A})$, then

$$\begin{aligned} Re < \mathscr{A}f, f >= & -(\frac{\nu_1}{2}+\nu_2\nu_3-\frac{\nu_3^2}{2}+\nu_4)|f(0)|^2 \\ & -\frac{1}{2}|f'(L)|^2-\nu_2\|f'\|^2+\nu_0\|f\|^2 \\ \leqslant & (\nu_0-\frac{1}{4L^2}\nu_2)\|f\|^2 \\ < & 0. \end{aligned} \tag{46}$$

Hence $\mathscr{A}$ is dissipative in $\mathbf{H}$, and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathbf{H}$ by the Lumer-Philips theorem.

■

Theorem 1: For each $\lambda \in \sigma(\mathscr{A})$, $Re\lambda < 0$. $\mathscr{A}$ generates an exponentially stable C_0-semigroup on $\mathbf{H}$. For any initial value $w(\cdot,0) \in \mathbf{H}$, there exists a unique (mild) solution to $(16)-(19)$ such that

$$w(\cdot,t) \in C([0,\infty);\mathbf{H}), \tag{47}$$

and there exists a positive constant ρ such that

$$\|w(\cdot,t)\| \leqslant e^{-\rho t}\|w(\cdot,0)\|. \tag{48}$$

Moreover, if $w(\cdot,0) \in D(\mathscr{A})$, then

$$w(\cdot,t) \in C^1([0,\infty);\mathbf{H}) \tag{49}$$

is the classical solution to $(16)-(19)$.

Proof: From the proof of Lemma 2, we have

$$Re < \mathscr{A}f, f > \leqslant -\rho\|f\|^2, \ \forall f \in D(\mathscr{A}), \tag{50}$$

where

$$\rho = \frac{1}{4L^2}\nu_2 - \nu_0 > 0. \tag{51}$$

Define a Lyapunov function

$$L(t) = \frac{1}{2}\|w(\cdot,t)\|^2, \tag{52}$$

then we can get

$$\dot{L}(t) \leqslant -2\rho L(t), \tag{53}$$

and thus

$$L(t) \leqslant L(0)e^{-2\rho t}. \tag{54}$$

Since $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$, this semigroup must be exponentially stable. ■

Remark 3: ρ is a lower bound estimate of exponential decay rate, which can be arbitrarily large by choosing ν_0 small enough. As can be seen from Appendix , it's not necessarily equal to decay rate.

IV. STATE FEEDBACK CONTROLLER

A transformation $v \mapsto w$ is to be seeked to transform the class of control systems $(6)-(9)$ into the exponentially stable target system $(16)-(19)$, and it's postulated in the following form

$$w(x,t) = v(x,t) - \int_0^x \kappa(x,y)v(y,t)dy, \tag{55}$$

where the gain function $\kappa(x,y) \in \mathbb{R}$ is to be determined.

Choose

$$\nu_1 = \mu_1, \ \nu_2 = \mu_2, \tag{56}$$

then a sufficient condition for $(16)-(18)$ to hold is that $\kappa(x,y)$ satisfies

$$\begin{aligned} & \kappa_{xxx}(x,y)+\kappa_{yyy}(x,y)+\mu_2(\kappa_{xx}(x,y)-\kappa_{yy}(x,y)) \\ & \quad +\mu_1(\kappa_x(x,y)+\kappa_y(x,y)) = (\mu_0-\nu_0)\kappa(x,y) \end{aligned} \tag{57}$$

$$\kappa(x,x) = \mu_3 - \nu_3 \tag{58}$$

$$\kappa_x(x,x) = \frac{\nu_0-\mu_0}{3}x - (\mu_3-\nu_3)\mu_3 + \mu_4 - \nu_4 \tag{59}$$

$$\kappa_{yy}(x,0) - (\mu_2+\mu_3)\kappa_y(x,0) + (\mu_1+\mu_2\mu_3+\mu_4)\,\kappa(x,0) = 0. \tag{60}$$

Let

$$\kappa(x,y) = p(x,y)e^{c(x-y)}, \ p(x,y) = G(\xi,\eta), \tag{61}$$

where

$$c = -\frac{\mu_2+\mu_3}{2}, \ \xi = x+y, \ \eta = x-y, \tag{62}$$

then

$$\begin{aligned} & 2G_{\xi\xi\xi}(\xi,\eta)+6G_{\xi\eta\eta}(\xi,\eta)-2(\mu_2+3\mu_3)G_{\xi\eta}(\xi,\eta) \\ & \quad +2\left(\mu_1-\frac{(\mu_2+\mu_3)(\mu_2-3\mu_3)}{4}\right)G_\xi(\xi,\eta) \\ & \quad = (\mu_0-\nu_0)G(\xi,\eta) \end{aligned} \tag{63}$$

$$G(\xi,0) = \mu_3 - \nu_3 \tag{64}$$

$$G_\eta(\xi,0) = \frac{\nu_0-\mu_0}{6}\xi + (\mu_3-\nu_3)\frac{\mu_2-\mu_3}{2} + \mu_4 - \nu_4 \tag{65}$$

$$\begin{aligned} & G_{\xi\xi}(\xi,\xi)-2G_{\xi\eta}(\xi,\xi)+G_{\eta\eta}(\xi,\xi) \\ & \quad +\left(\mu_1-\frac{(\mu_2-\mu_3)^2}{4}+\mu_4\right)G(\xi,\xi) = 0. \end{aligned} \tag{66}$$

By a lengthy calculation, an integral equation can be obtained:

$$G(\xi,\eta) = G^0(\xi,\eta) + F[G](\xi,\eta). \tag{67}$$

Here

$$\begin{aligned} G^0(\xi,\eta) = & \frac{\nu_0-\mu_0}{6}\eta(\xi-\eta) + Je^{E\eta}\begin{pmatrix}1\\0\end{pmatrix} \\ & +\frac{2}{3}(\nu_0-\mu_0)\int_0^\eta \begin{pmatrix}0 & 1\end{pmatrix} e^{E(\eta-\sigma)}\begin{pmatrix}1\\0\end{pmatrix} d\sigma \end{aligned} \tag{68}$$

and

$$F[G](\xi,\eta)=\int_{\eta}^{\xi}\int_{0}^{\eta}\int_{0}^{\tau}(d_1G(s,t)+d_2G_s(s,t)+d_3G_{st}(s,t)+d_4G_{sss}(s,t))\,dtd\tau ds+4\int_{0}^{\eta}\int_{0}^{\sigma}(d_1G(\sigma,t)+d_2G_{\sigma}(\sigma,t)+d_3G_{\sigma t}(\sigma,t)+d_4G_{\sigma\sigma\sigma}(\sigma,t))\,dt\times\begin{pmatrix}0 & 1\end{pmatrix}e^{E(\eta-\sigma)}\begin{pmatrix}1\\0\end{pmatrix}d\sigma, \tag{69}$$

where

$$E=\begin{pmatrix}0 & m\\ 1 & 0\end{pmatrix} \tag{70}$$

$$J=\begin{pmatrix}\mu_3-\nu_3 & (\mu_3-\nu_3)\frac{\mu_2-\mu_3}{2}+\mu_4-\nu_4\end{pmatrix} \tag{71}$$

$$m=-\left(\mu_1-\frac{(\mu_2-\mu_3)^2}{4}+\mu_4\right) \tag{72}$$

and

$$d_1=\frac{1}{6}(\mu_0-\nu_0) \tag{73}$$

$$d_2=-\frac{1}{3}\left(\mu_1-\frac{(\mu_2+\mu_3)(\mu_2-3\mu_3)}{4}\right) \tag{74}$$

$$d_3=\frac{1}{3}(\mu_2+3\mu_3) \tag{75}$$

$$d_4=-\frac{1}{3}. \tag{76}$$

Let

$$G^{n+1}(\xi,\eta)=F[G^n(\xi,\eta)],n=0,1,2,\cdots, \tag{77}$$

then

$$G(\xi,\eta)=\sum_{n=0}^{\infty}G^n(\xi,\eta). \tag{78}$$

Denote

$$e_1=|d_1|,\ e_2=|d_2|,\ e_3=|d_3|,\ e_4=|d_4| \tag{79}$$

and

$$M=\frac{2}{3}|\mu_0-\nu_0|+\sup_{0\leqslant\eta\leqslant1}\left|Je^{E\eta}\begin{pmatrix}1\\0\end{pmatrix}+\frac{2}{3}(\nu_0-\mu_0)\int_0^{\eta}\begin{pmatrix}0 & 1\end{pmatrix}e^{E(\eta-\sigma)}\begin{pmatrix}1\\0\end{pmatrix}d\sigma\right|+2\sup_{0\leqslant\eta\leqslant1}\left|Je^{E\eta}\begin{pmatrix}0\\1\end{pmatrix}+\frac{2}{3}(\nu_0-\mu_0)\int_0^{\eta}\begin{pmatrix}0 & 1\end{pmatrix}e^{E(\eta-\sigma)}\begin{pmatrix}0\\1\end{pmatrix}d\sigma\right| \tag{80}$$

$$N=4(e_1+e_2+e_3+e_4)\times\left(1+\sup_{0\leqslant\eta\leqslant1}\left|\begin{pmatrix}0 & 1\end{pmatrix}e^{E(\eta-\sigma)}\begin{pmatrix}1\\0\end{pmatrix}\right|+\sup_{0\leqslant\eta\leqslant1}\left|\begin{pmatrix}0 & 1\end{pmatrix}e^{E(\eta-\sigma)}\begin{pmatrix}0\\1\end{pmatrix}\right|\right), \tag{81}$$

then we can get

$$|G^1(\xi,\eta)|\leqslant MN(\xi+\eta) \tag{82}$$

$$|G^1_{\eta}(\xi,\eta)|\leqslant MN \tag{83}$$

$$|G^1_{\xi}(\xi,\eta)|\leqslant MN \tag{84}$$

$$|G^1_{\xi\eta}(\xi,\eta)|\leqslant MN \tag{85}$$

$$|G^1_{\xi\xi}(\xi,\eta)|\leqslant MN \tag{86}$$

$$|G^1_{\xi\xi\eta}(\xi,\eta)|\leqslant MN \tag{87}$$

$$|G^1_{\xi\xi\xi}(\xi,\eta)|=0. \tag{88}$$

Moreover, by mathematical induction, it can be proved that for $n\geqslant1$,

$$|G^n(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^n}{n!} \tag{89}$$

$$|G^n_{\eta}(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \tag{90}$$

$$|G^n_{\xi}(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \tag{91}$$

$$|G^n_{\underbrace{\xi\xi\cdots\xi}_{m-1}\eta}(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \tag{92}$$

$$|G^n_{\underbrace{\xi\xi\cdots\xi}_{m}}(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \tag{93}$$

$$|G^n_{\underbrace{\xi\xi\cdots\xi}_{n+1}\eta}(\xi,\eta)|\leqslant MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!} \tag{94}$$

$$|G^n_{\underbrace{\xi\xi\cdots\xi}_{n+2}}(\xi,\eta)|=0, \tag{95}$$

where

$$2\leqslant m\leqslant n+1. \tag{96}$$

From (68) – (69), we can get that $G^n(\xi,\eta)$ is C^3. Therefore, $G(\xi,\eta)=\sum_{n=0}^{\infty}G^n(\xi,\eta)$ converges absolutely and uniformly, and $G(\xi,\eta)$ is C^3 which has a bound

$$|G(\xi,\eta)|\leqslant Me^{N(\xi+\eta)}. \tag{97}$$

Since we have found the function $G(\xi,\eta)$, existence of function $p(x,y)$ and kernel $\kappa(x,y)$ is obtained. Moreover, since the transformation (55) is continuous, there exists a positive constant C_κ such that

$$\|w\|\leqslant C_\kappa\|v\|. \tag{98}$$

The backstepping transformation (55) is invertible, and inverse transformation $w\mapsto v$ can also be postulated as follows:

$$v(x,t)=w(x,t)+\int_0^x\iota(x,y)w(y,t)dy, \tag{99}$$

which satisfies

$$\iota_{xxx}(x,y)+\iota_{yyy}(x,y)+\mu_2(\iota_{xx}(x,y)-\iota_{yy}(x,y)) + \mu_1(\iota_x(x,y)+\iota_y(x,y))=(\nu_0-\mu_0)\iota(x,y) \quad (100)$$

$$\iota(x,x)=\mu_3-\nu_3 \quad (101)$$

$$\iota_x(x,x)=\frac{\nu_0-\mu_0}{3}x-(\mu_3-\nu_3)\nu_3+\mu_4-\nu_4 \quad (102)$$

$$\iota_{yy}(x,0)-(\mu_2+\nu_3)\iota_y(x,0)+(\mu_1+\mu_2\nu_3+\nu_4)\iota(x,0)=0. \quad (103)$$

Similar results about existence and regularity of the kernel $\iota(x,y)$ can be proved in a similar way as proving for kernel $\kappa(x,y)$. Moreover, the inverse transformation is also continuous, and thus there exists a positive constant C_ι such that

$$\|v\| \leqslant C_\iota \|w\|. \quad (104)$$

Then from $(5),(48),(98),(104)$, there exists a constant C_ε such that

$$\|u(\cdot,t)\| \leqslant C_\varepsilon C_\iota C_\kappa e^{-\rho t}\|u(\cdot,0)\|, \quad (105)$$

which proves exponential decay for the class of closed-loop control systems $(1)-(4)$ with controllers

$$U(t)=\int_0^L \kappa(L,y)u(y,t)e^{\varepsilon(y-L)}dy. \quad (106)$$

Theorem 2: For any initial value $u(\cdot,0)\in\mathbf{H}$, there exists a unique (mild) solution to the closed-loop system $(1)-(4)$ with (106) such that

$$u(\cdot,t)\in C([0,\infty);\mathbf{H}), \quad (107)$$

and there exists positive constants M_u,ρ such that

$$\|u(\cdot,t)\| \leqslant M_u e^{-\rho t}\|u(\cdot,0)\|. \quad (108)$$

Moreover, if $u(\cdot,0)$ satisfies boundary compatibility condition, then

$$u(\cdot,t)\in C^1([0,\infty);\mathbf{H}) \quad (109)$$

is the classical solution.

V. An Example

Consider the following subclass of control systems as an example:

$$u_t(x,t)=u_{xxx}(x,t)+\lambda_0 u(x,t) \quad (110)$$

$$u_x(0,t)=0 \quad (111)$$

$$u_{xx}(0,t)=0 \quad (112)$$

$$u(1,t)=U(t). \quad (113)$$

Choose $\varepsilon=0$, that is, $v(x,t)=u(x,t)$, and set the target system as follows:

$$w_t(x,t)=w_{xxx}(x,t)+\nu_0 w(x,t) \quad (114)$$

$$w_x(0,t)=0 \quad (115)$$

$$w_{xx}(0,t)=0 \quad (116)$$

$$w(1,t)=0. \quad (117)$$

Through spectrum analysis and some calculation, we get that, for $\lambda_0>6.3297$, the open-loop systems $(110)-(113)$ (with $U(t)=0$) have eigenvalues on RHS of the complex plane and thus are unstable. However, by choosing $\nu_0<6.3297$, all eigenvalues of target systems are on LHS of the complex plane (see, e.g., TABLE 1) and thus the equivalent closed-loop control systems are asymptotically stable. What's more, for $\nu_0<0$, we have proved that they're exponentially stable, and the exponential decay rate can be arbitrarily large by choosing ν_0 to be small enough.

Real parts of first 7 eig.	uncontrolled system with $\lambda_0=100$	closed-loop system with $\nu_0=-100$
1st eig.	93.6703	-106.3297
2nd eig.	-61.1000	-261.1000
3rd eig.	-645.9000	-845.9000
4th eig.	-1.9467×10^3	-2.1467×10^3
5th eig.	-4.2501×10^3	-4.4501×10^3
6th eig.	-7.8423×10^3	-8.0423×10^3
7th eig.	-1.3010×10^4	-1.3210×10^4

TABLE I

Real parts of first seven eigenvalues

Remark 4: The eigenvalues of $(110)-(113)$ (with $U(t)=0$) and $(114)-(117)$ are $(ln\theta)^3+\lambda_0$ and $(ln\theta)^3+\nu_0$ respectively, where θ are roots of the following equation:

$$\theta+\theta^{\omega}+\theta^{\omega^2}=0 \quad (118)$$

That is, eigenvalues of target systems are open-loop eigenvalues shifted to the left in the complex plane by the same distance $\lambda_0-\nu_0$.

For the kernel function, first we have

$$G_0(\xi,\eta)=\frac{\nu_0-\mu_0}{6}\eta(\xi+\eta). \quad (119)$$

Then, by performing some lengthy calculations, we get the following formula:

$$G^k(\xi,\eta)=\sum_{i=0}^{[\frac{k}{3}]}\Big(a_{i,0,k}\eta^{3k+2-3i} + \sum_{j=1}^{k+1-3i} a_{i,j,k}\eta^{3k+2-j-3i}(\xi^j-\eta^j)\Big) = \sum_{i=0}^{[\frac{k}{3}]}\eta^{3k+2-3i}\left(\sum_{j=0}^{k+1-3i} b_{i,j,k}(\frac{\xi}{\eta})^j\right) \quad (120)$$

for $k\geqslant 1$, where all coefficients $a_{i,0,k},a_{i,j,k},b_{i,j,k}$ are constants and $[x]$ denotes the integer not larger than x.

VI. Conclusion and Future Work

In this paper, backstepping boundary controllers are designed for a class of linearized KdV systems with possible anti-diffusion. The target systems considered can be exponentially stable with arbitrary decay rate. Since the backstepping transformation is invertible, same properties hold for the resulting closed-loop control system.

For future work, we are to consider control design for cascaded/coupled KdV-ODE systems with possible anti-diffusion, such as

$$\begin{aligned}\dot{X}(t) =& AX(t)+Bu(0,t) \quad (121)\\ u_t(x,t) =& u_{xxx}(x,t)+\lambda_2 u_{xx}(x,t)+\lambda_1 u_x(x,t)\\ &+\lambda_0 u(x,t), x\in(0,L) \quad (122)\\ u_x(0,t) =& \lambda_3 u(0,t)+CX(t) \quad (123)\\ u_{xx}(0,t) =& \lambda_4 u(0,t) \quad (124)\\ u(L,t) =& U(t). \quad (125)\end{aligned}$$

Another problem which might bring some challenges is to derive optimal decay rates for the target systems and resulting closed-loop control systems.

Appendix

If choosing $\nu_3=\nu_4=0$, then for the class of target systems $(16)-(19)$ with

$$\nu_1 \geqslant 0,\ \nu_2 \geqslant 0,\ \nu_0 \leqslant \frac{1}{4L^2}\nu_2, \quad (126)$$

the following lemma holds.

Lemma 3: For each $\lambda\in\sigma(\mathscr{A})$, $Re\lambda<0$. Moreover, $\mathscr{A}$ generates an asymptotically stable C_0-semigroup on $\mathbf{H}$.

Proof: Following the proof of Lemma 2, we can get that for each $\lambda\in\sigma(\mathscr{A})$, $Re\lambda\leqslant 0$. Let $\lambda\in\sigma(\mathscr{A})$ be on the imaginary axis and $f\in D(\mathscr{A})$ be its associated eigenfunction of $\mathscr{A}$, then we have $Re<\mathscr{A}f,f>=0$, hence, $f'(L)=0$, $\nu_0=\nu_1=\nu_2=0$. That is, there exist $y(x)\in\mathbf{H}^3(0,L)\backslash\{0\}$ and λ on the imaginary axis such that

$$y'''-\lambda y=0, x\in(0,L) \quad (127)$$

$$y'(0)=y''(0)=y(L)=y'(L)=0. \quad (128)$$

Denote by $z\in\mathbf{H}^3(\mathbb{R})$ its prolongation by 0, then

$$z'''-\lambda z=y(0)\delta_0''-y''(L)\delta_L \ in\ \mathscr{D}'(\mathbb{R}), \quad (129)$$

where δ_{x_0} denotes the Dirac measure at x_0. This is equivalent to the existence of complex numbers ϕ,ψ,λ (with $\phi\neq 0$, $\psi\neq 0$) and a function $z\in\mathbf{H}^3(\mathbb{R})$ with compact support in $[-L,L]$ such that

$$z'''-\lambda z=\phi\delta_0''-\psi\delta_L \ in\ \mathscr{D}'(\mathbb{R}). \quad (130)$$

Take Fourier transformation, then

$$\left((i\xi)^3-\lambda\right)\hat{z}(\xi)=\phi(i\xi)^2-\psi e^{-iL\xi} \ in\ \mathscr{D}'(\mathbb{R}), \quad (131)$$

and (setting $\lambda=-ip^3$)

$$\hat{z}(\xi)=-i\frac{\phi\xi^2+\psi e^{-iL\xi}}{\xi^3-p^3}. \quad (132)$$

Thus, there exist $p\in\mathbb{C}$ and $(\phi,\psi)\in\mathbb{C}^2\backslash\{(x,y)|x\neq 0,y\neq 0\}$ such that

$$f(\xi):=\frac{\phi\xi^2+\psi e^{-iL\xi}}{\xi^3-p^3} \quad (133)$$

is an entire function in $\mathbb{C}$. Since the roots of ξ^3-p^3 are $p,\omega p,\omega^2 p$, this holds only if they are all also roots of $\phi\xi^2+\psi e^{-iL\xi}$. Then we have

$$e^{-iLp}=-\frac{\phi}{\psi}p^2 \quad (134)$$

$$e^{-iL\omega p}=-\frac{\phi}{\psi}\omega^2 p^2 \quad (135)$$

$$e^{-iL\omega^2 p}=-\frac{\phi}{\psi}\omega^4 p^2. \quad (136)$$

Substitute (134) into (135) and (136), multiply both sides of the resulting equations, then

$$p^2=-\frac{\psi}{\phi},-\omega\frac{\psi}{\phi}\ or\ -\omega^2\frac{\psi}{\phi}. \quad (137)$$

However, by substituting (137) into (135), we get contradictions for all three cases, which proves that for each $\lambda\in\sigma(\mathscr{A})$, $Re\lambda<0$. Moreover, since from (54),

$$L(t)\leqslant L(0), \quad (138)$$

then $\mathscr{A}$ generates an asymptotically stable C_0-semigroup on $\mathbf{H}$ by the Arendt-Batty-Lyubich-Phong theorem. ■

Remark 5: If furtherly choosing $\nu_0=\nu_1=\nu_2=0$, then the class of target systems $(16)-(19)$ has been proved to be exponentially stable in [12].

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