State and Disturbance Estimator for Unstable Reaction-Advection-Diffusion PDE with Boundary Disturbance *

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Abstract

We propose an asymptotically convergent state and disturbance estimator for a class of reaction-advection-diffusion PDEs where the disturbance is anti-collocated with control input, using measurements at both boundaries. Two auxiliary systems are designed for disturbance estimation, adopting the output error injection method, where the disturbance estimator is determined by the plant model structure and measured signals. A sufficient condition on the reaction coefficient is derived for which the disturbance estimator achieves asymptotic convergence to the true value. All states in the disturbance estimator are proved to be bounded using Lyapunov stability theory. We further propose a state estimator using the backstepping technique, by injecting the disturbance estimation signal into the state observer.

1 Introduction

This paper considers a general reaction-advection-diffusion partial differential equation (PDE) with unknown boundary disturbances, which can be utilized to describe a variety of systems such as thermal/fluid flows [4], electrochemistry [5], and structural acoustics [6], with uncertain flux at one end. The objective of this paper is to estimate the disturbance at the boundary, in order to attenuate the effect of disturbance in the feedback controller design. With the disturbance estimation signal, a state estimator is also proposed.

In the past few decades, the boundary control and estimation of PDE systems has gained significant research attention, due to their high-fidelity model accuracy in describing many processes. When uncertainties enter the PDE system through the boundaries or in-domain dynamics, there generally have been three types of methods developed to tackle such issue: adaptive control [8], sliding mode control [7, 9], and active disturbance control [10, 11]. See [7] for an excellent review.

The active disturbance rejection control (ADRC) method was initially introduced by Han [12], which has been proven to be effective in dealing with disturbances in PDE systems. A crucial step in ADRC is to estimate the time-varying disturbance using available boundary measurements. The convergence problem of ADRC was solved in [13], and this approach has been widely applied to disturbance attenuation in feedback controller design in PDE systems. For instance, a boundary output feedback stabilization for a one-dimensional anti-stable wave equation with control matched disturbance is examined in [11]. A disturbance estimator for a wave PDE on a time-varying domain is studied in [14], and an output feedback controller is further designed utilizing the disturbance estimates [10]. The output feedback stabilization for an unstable wave equation with general boundary measurement disturbance is introduced in [15]. The application of ADRC on the unstable heat equation with boundary uncertainties is presented in [7], as well as the sliding mode controller design. In [3], stabilization of an unstable 1-D heat equation with boundary uncertainty and external disturbance is achieved by designing an unknown input type state observer.

In this paper, we design a combined disturbance and state estimator for an unstable reaction-advection-diffusion PDE with boundary disturbance, by adopting a similar methodology from [3, 14]. The contribution of this paper lies in

- *Designing a disturbance estimator for boundary disturbance in an unstable reaction-advection-diffusion PDE system, and derive a sufficient condition on the reaction coefficient, for which the disturbance estimator achieves asymptotic convergence.*

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- *Proposing an asymptotically convergent state estimator for the unstable reaction-advection-diffusion PDE using the estimated disturbance signal, adopting the backstepping technique.*

This paper is organized as follows. Section 2 discusses the problem set-up, well-posedness of the plant model. The disturbance estimator and the corresponding convergence analysis are presented in Section 3. Section 4 presents the state estimator design using the estimated disturbance signal, by employing the backstepping method. Section 5 provides a numerical simulation to visualize the performance of the proposed estimators. Section 6 summarizes and concludes the paper.

Notation. Throughout the manuscript, $u(x,t)$ denotes the state variable with the dependence on space variable x and time variable t. The x and t subscripts represent partial derivatives with respect to the notated variable: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, and $u_{xx} = \partial^2 u/\partial x^2$. The dot symbol denotes derivative with respect to time t, e.g. $\dot{T} = \mathrm{d}T/\mathrm{d}t$, and the prime symbol represents derivative with respect to space x, e.g. $X' = \mathrm{d}X/\mathrm{d}x$. The $L^2(0,1)$ spatial norm is defined as $\|u(\cdot,t)\| = \sqrt{\int_0^1 u^2(x,t)\mathrm{d}x}$.

2 Problem Specification.

We consider the following reaction-advection-diffusion PDE with boundary disturbance, where the disturbance is anti-collocated with the applied control input:

$$z_t(x,t) = z_{xx}(x,t) + bz_x(x,t) + \lambda_0 z(x,t), \tag{2.1}$$

$$z_x(0,t) = q_0 z(0,t) + d_0(t), \tag{2.2}$$

$$z_x(1,t) = Q(t), \tag{2.3}$$

$$z(x,0) = z_0(x), \tag{2.4}$$

$$y(t) = \{z(0,t), z(1,t)\}, \tag{2.5}$$

where b, λ_0, and q_0 are constants, $d_0(t)$ represents the boundary disturbance, and $Q(t)$ denotes the control input. The signals at both boundaries are measured. The following change of variables (gauge transformation) [2]:

$$u(x,t) = z(x,t)e^{\frac{b}{2}x}, \tag{2.6}$$

transforms the system (2.1)-(2.5) to (2.7)-(2.11), with coefficients, disturbance, and control input mapped accordingly, as follows,

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{2.7}$$

$$u_x(0,t) = qu(0,t) + d(t), \tag{2.8}$$

$$u_x(1,t) = U(t), \tag{2.9}$$

$$u(x,0) = u_0(x). \tag{2.10}$$

$$y_m(t) = \{u(0,t), u(1,t)\}. \tag{2.11}$$

Symbol $d(t)$ represents the disturbance on the heat flux at one boundary, $U(t)$ denotes the known control input, and λ and q are constants.

Assumption 1. *The disturbance $d(t) \in \mathbb{R}$ is upper and lower bounded:*

$$\left|d(t)\right| \le \overline{d}, \quad \forall t \in [0,\infty), \tag{2.12}$$

where $\overline{d}$ is an unknown positive number.

The analysis in this paper is based on system (2.7)-(2.11). Our objective is to estimate the disturbance $d(t)$ as well as the state $u(x,t)$ by utilizing the boundary measurements $y_m(t)$.

Theorem 2.1. *The linear boundary value problem (BVP) (2.7)-(2.10) is well-posed with initial data $u_0(\cdot) \in L^2(0,1)$, provided that the disturbance $d(t)$ and control input $U(t)$ are bounded.*

We utilize Lemma A.1 [1] (in Appendix) to prove Theorem 2.1.

Proof. Define the operator

$$(\mathcal{L}u)(x,t) = -u_{xx}(x,t) - \lambda u(x,t), \quad (x,t) \in (0,1)\times[0,\infty). \tag{2.13}$$

Similarly, define the boundary condition operator

$$(\mathcal{B}u)(x,t) = \begin{cases} u_x(0,t) - qu(0,t) & x = 0, t \in [0,\infty) \\ u_x(1,t) & x = 1, t \in [0,\infty) \end{cases} \tag{2.14}$$

which allows the boundary condition to be expressed as

$$(\mathcal{B}u)(x,t) = h(x,t), \tag{2.15}$$

where $h(0,t) = d(t)$ and $h(1,t) = U(t)$. Then, by defining

$$(\mathcal{H}u) = \begin{cases} u_t(x,t) + (\mathcal{L}u)(x,t) & (x,t) \in (0,1)\times[0,\infty) \\ (\mathcal{B}u)(x,t) & (x,t) \in \{0,1\}\times[0,\infty) \\ u(x,0) & x \in [0,1], t = 0 \end{cases} \tag{2.16}$$

and

$$\mathcal{F}(x,t) = \begin{cases} 0 & (x,t) \in (0,1)\times[0,\infty) \\ h(x,t) & (x,t) \in \{0,1\}\times[0,\infty) \\ u_0(x) & x \in [0,1], t = 0 \end{cases} \tag{2.17}$$

the BVP (2.7)-(2.10) can be written in the compact form:

$$(\mathcal{H}u)(x,t) = \mathcal{F}(x,t). \tag{2.18}$$

The operator $\mathcal{H}$ is linear. The inverse monotonicity of $\mathcal{H}$ can be confirmed by contradiction. Moreover, a non-negative comparison function $\phi(x)$, $x \in [0,1]$, can be computed by constructing a low-degree polynomial, for example, $\phi(x) = Ax^2 + Bx + C$, and choose constants A, B, C to verify $\mathcal{H}\phi(x) \geq 1$ for $x \in [0,1]$. Then the well-posedness follows immediately from Lemma A.1, by choosing $\|\psi\|_u = \|\psi\|_\infty = \max_{(x,t)\in[0,1]\times[0,\infty)} \psi$, for $\psi \in L^2(0,1)$. Furthermore, from (A.4) we have that

$$\max_{x\in[0,1]} \left|u(x,t)\right| \leq \left[\max_{x\in[0,1]} \phi(x)\right] \cdot \max\{\|h(x,t)\|_\infty, \|u_0(x)\|_\infty\}, \tag{2.19}$$

which dictates that a bound on the magnitude of the solution has been determined.

REMARK 2.1. *The plant model dynamics (2.7)-(2.10) is unstable for sufficiently large λ and q. This motivates future ADRC design where the disturbance estimation is required to attenuate the actual disturbance.*

3 Disturbance Estimator Design

In this section, we detail the disturbance estimator design for system (2.7)-(2.10) using boundary measurement $y_m(t)$. We introduce the following auxiliary system:

$$\eta_t(x,t) = \eta_{xx}(x,t) + \lambda\eta(x,t), \tag{3.20}$$
$$\eta(0,t) = u(0,t) - \zeta(0,t), \tag{3.21}$$
$$\eta_x(1,t) = -\alpha\eta(1,t), \tag{3.22}$$

where $\zeta(x,t)$ satisfies the following system:

$$\zeta_t(x,t) = \zeta_{xx}(x,t) + \lambda\zeta(x,t), \tag{3.23}$$
$$\zeta_x(0,t) = qu(0,t), \tag{3.24}$$
$$\zeta_x(1,t) = U(t) + \alpha(u(1,t) - \zeta(1,t)). \tag{3.25}$$

Specifically, ζ system consists of a copy of the plant model (2.7)-(2.10) with the output error injection using the measurement of $u(1,t)$. The η system is completely determined by the measured signal $u(0,t)$ and the boundary value from the ζ system. We further define the estimate for the disturbance $\hat{d}(t)$ to be

$$\hat{d}(t) = \eta_x(0,t). \tag{3.26}$$

The system (3.20)-(3.22), (3.23)-(3.25), together with (3.26), is the *disturbance estimator*. The constant $\alpha > 0$ is to be determined such that the disturbance estimate $\hat{d}(t)$ reconstructs the actual disturbance $d(t)$ asymptotically.

3.1 Convergence of Disturbance Estimator

Define the variable $y(x,t) = u(x,t) - \zeta(x,t)$, which satisfies the system:

$$y_t(x,t) = y_{xx}(x,t) + \lambda y(x,t), \tag{3.27}$$
$$y_x(0,t) = d(t), \tag{3.28}$$
$$y_x(1,t) = -\alpha y(1,t), \tag{3.29}$$

and we also define $\tilde{w} = y - \eta$, which verifies

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + \lambda\tilde{w}(x,t), \tag{3.30}$$
$$\tilde{w}(0,t) = 0, \tag{3.31}$$
$$\tilde{w}_x(1,t) = -\alpha\tilde{w}(1,t). \tag{3.32}$$

The purpose of $\tilde{w}$ system is that the disturbance estimation error $\tilde{d}(t) = d(t) - \hat{d}(t)$ can be expressed by the boundary signal of $\tilde{w}$, as follows,

$$\tilde{d} = d - \hat{d} = y_x(0,t) - \eta_x(0,t) = \tilde{w}_x(0,t). \tag{3.33}$$

Thus, the convergence analysis of disturbance estimation error $\tilde{d}(t)$ is equivalent to the convergence of $\tilde{w}_x(0,t)$.

REMARK 3.1. *The systems (3.27)-(3.29) and (3.30)-(3.32) are well-posed. The structure of the proof is analogous to Theorem 2.1 using Lemma A.1.*

From now on, we aim to determine the values of parameter λ such that there always exists a tuning parameter $\alpha > 0$ for which $\tilde{w}(x,t)$ and $\tilde{w}(x,t)$ converge to zero as $t \to \infty$ in the sense of L^2 norm. Prior to presenting the main theorem, we require a few lemmas. The next lemma is the extension of the well-known Poincaré Inequality [2, 16]. For the reader's convenience, we provide a sketch of the proof.

LEMMA 3.1. *For any function $\tilde{w}(x,t)$ with $x \in [0,1]$ and $t \in [0,\infty)$, that is twice continuously differentiable on $x \in [0,1]$,*

$$\|\tilde{w}_x(x,t)\|^2 \leq 2\tilde{w}_x^2(1,t) + 4\|\tilde{w}_{xx}(x,t)\|^2. \tag{3.34}$$

Proof.

$$\int_0^1 \tilde{w}_x^2 \mathrm{d}x = x\tilde{w}_x^2\Big|_0^1 - 2\int_0^1 x\tilde{w}_x\tilde{w}_{xx}\mathrm{d}x \leq \tilde{w}_x^2(1) + \frac{1}{2}\int_0^1 \tilde{w}_x^2\mathrm{d}x + 2\int_0^1 x^2\tilde{w}_{xx}^2\mathrm{d}x, \tag{3.35}$$

where Young's Inequality has been used. Therefore,

$$\int_0^1 \tilde{w}_x^2\mathrm{d}x \leq 2\tilde{w}_x^2(1) + 4\int_0^1 \tilde{w}_{xx}^2\mathrm{d}x. \tag{3.36}$$

LEMMA 3.2. *System (3.30)-(3.32) admits an unique solution $\tilde{w}(x,t)$ which satisfies*

$$\|\tilde{w}_{xx}(\cdot,t)\| \leq \|\tilde{w}_{xx}(\cdot,0)\| L e^{-\Omega t}, t \geq 0, \tag{3.37}$$

where L and Ω are positive constants, given that $\lambda < x_0^2$, where x_0 is the smallest positive solution to

$$\tan(x) = -\frac{x}{\alpha}. \tag{3.38}$$

The proof of Lemma 3.2 is omitted here. The readers may refer to Lemma 3.1 in [3] for details.

Now we present the following lemma describing the stability results for the system (3.30)-(3.32).

LEMMA 3.3. *For any initial data $\tilde{w}_0(\cdot) \in L^2(0,1)$, and $\lambda < 3 - 2\sqrt{2}$, there exists a constant $\alpha > 0$ such that $\tilde{w}(x,t)$ in the system (3.30)-(3.32) is asymptotically stable for all $x \in [0,1]$. Moreover, $\tilde{w}_x(0,t)$ converges to zero as $t \to \infty$.*

Proof. Consider the Lyapunov functional

$$V(t) = \frac{\alpha}{2}\tilde{w}^2(1,t) + \frac{1}{2}\|\tilde{w}\|^2 + \frac{1}{2}\|\tilde{w}_x\|^2. \tag{3.39}$$

The time derivative of the Lyapunov functional $V(t)$ along the trajectory of $\tilde{w}(x,t)$ is

$$\begin{aligned}
\dot{V}(t) =& \alpha\tilde{w}(1)\tilde{w}_t(1) + \int_0^1 \tilde{w}\tilde{w}_t \mathrm{d}x + \int_0^1 \tilde{w}_x\tilde{w}_{xt}\mathrm{d}x \\
=& \alpha\tilde{w}(1)\tilde{w}_t(1) + \tilde{w}(x)\tilde{w}_x(x)\Big|_0^1 - \int_0^1 \tilde{w}_x^2 \mathrm{d}x \\
&+ \lambda\int_0^1 \tilde{w}^2\mathrm{d}x + \tilde{w}_t(x)\tilde{w}_x(x)\Big|_0^1 - \int_0^1 \tilde{w}_t\tilde{w}_{xx}\mathrm{d}x \\
=& -\alpha\tilde{w}^2(1) - \|\tilde{w}_x\|^2 + \lambda\|\tilde{w}\|^2 - \|\tilde{w}_{xx}\|^2 \\
&- \lambda\int_0^1 \tilde{w}\tilde{w}_{xx}\mathrm{d}x \\
=& -\alpha\tilde{w}^2(1) - \|\tilde{w}_x\|^2 + \lambda\|\tilde{w}\|^2 - \|\tilde{w}_{xx}\|^2 \\
&+ \lambda\alpha\tilde{w}^2(1) + \lambda\|\tilde{w}_x\|^2 \\
=& -\alpha(1-\lambda)\tilde{w}^2(1) + \lambda\|\tilde{w}\|^2 - (1-\lambda)\|\tilde{w}_x\|^2 \\
&- \|\tilde{w}_{xx}\|^2 \\
\leq& -\alpha(1-\lambda)\tilde{w}^2(1) + \lambda\|\tilde{w}\|^2 - (1-\lambda)\|\tilde{w}_x\|^2 \\
&+ \frac{1}{2}\tilde{w}_x^2(1) - \frac{1}{4}\|\tilde{w}_x\|^2 \\
=& -\left[\alpha(1-\lambda) - \frac{\alpha^2}{2}\right]\tilde{w}^2(1) + \lambda\|\tilde{w}\|^2 \\
&- \left(\frac{5}{4} - \lambda\right)\|\tilde{w}_x\|^2,
\end{aligned} \tag{3.40}$$

where integration by parts has been utilized multiple times, and Lemma 3.1 is applied in the last inequality. Observing the last line of (3.40), λ clearly has a significant impact on the sign of $\dot{V}(t)$. A sufficient condition posed on λ for which there always exists a $\alpha > 0$ such that $V(t)$ is asymptotically stable is to be determined.

We first require $\lambda < 5/4$ in view of the last term involving $\|\tilde{w}_x\|$ in (3.40), and introduce two positive constants p_1 and p_2 as follows,

$$p_1 + p_2 = \frac{5}{4} - \lambda, \text{ and } p_1, p_2 > 0, \tag{3.41}$$

so that (3.40) becomes

$$\begin{aligned}
\dot{V}(t) =& -\left[\alpha(1-\lambda) - \frac{\alpha^2}{2}\right]\tilde{w}^2(1) + \lambda\|\tilde{w}\|^2 \\
&- p_1\|\tilde{w}_x\|^2 - p_2\|\tilde{w}_x\|^2 \\
\leq& -\left[(1-\lambda) - \frac{\alpha}{2} - \frac{p_1}{2\alpha}\right]\alpha\tilde{w}^2(1) \\
&- \left(\frac{p_1}{4} - \lambda\right)\|\tilde{w}\|^2 - p_2\|\tilde{w}_x\|^2.
\end{aligned} \tag{3.42}$$

where the Poincaré Inequality is used. According to (3.42), if there exists a p_1 such that there always exists a $\alpha > 0$ so that the following conditions hold:

$$\begin{aligned}
&(1-\lambda) - \frac{\alpha}{2} - \frac{p_1}{2\alpha} > 0, \quad \frac{p_1}{4} - \lambda > 0, \\
&\frac{5}{4} - \lambda - p_1 > 0, \quad \text{and } p_1 > 0,
\end{aligned} \tag{3.43}$$

then the Lyapunov functional $V(t)$ decays exponentially with decaying rate β:

$$\dot{V} \leq -\beta V. \tag{3.44}$$

where $\beta > 0$ is defined by

$$\beta = \min\left\{(1-\lambda) - \frac{\alpha}{2} - \frac{p_1}{2\alpha}, \frac{p_1}{4} - \lambda, \frac{5}{4} - \lambda - p_1\right\}. \tag{3.45}$$

In order for p_1 to be well-defined by (3.43) for some α, we must enforce

$$\max\{0, 4\lambda\} < p_1 < \min\left\{2\alpha(1-\lambda) - \alpha^2, \frac{5}{4} - \lambda\right\}. \tag{3.46}$$

We solve for the values of λ such that there always exists a $\alpha > 0$ so that (3.46) holds. We consider two cases: $\lambda \leq 0$ and $\lambda > 0$.

Case 1: $\lambda \leq 0$. In this case, the left hand side (LHS) of (3.46) is LHS $= \max\{0, 4\lambda\} = 0$. On the right hand side (RHS) of (3.46), we have $5/4 - \lambda > 0$. Let $f(\alpha) = 2\alpha(1-\lambda) - \alpha^2$, and $f(\alpha)$ takes its peak value $f_{\max} = (1-\lambda)^2 > 0$ at $\alpha = (1-\lambda) > 0$, which means that there always exists a $\alpha > 0$ such that RHS > 0. Thus, there always exists a $p_1 > 0$ that satisfies (3.46) if $\lambda \leq 0$.

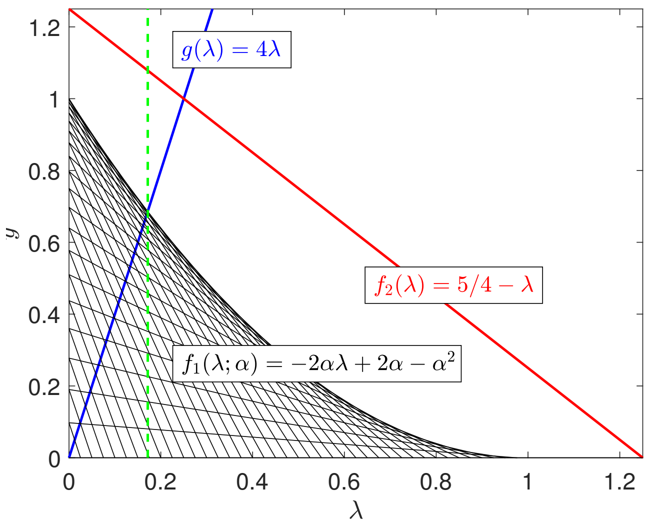

Figure 1: Visualization of analysis of sufficient condition for reaction coefficient λ.

Case 2: $\lambda > 0$. In this scenario, LHS $= 4\lambda$. Let $g(\lambda) = 4\lambda$, $f_1(\lambda;\alpha) = 2\alpha(1-\lambda) - \alpha^2$, and $f_2(\lambda) = 5/4 - \lambda$. Note that function f_1 is parametrized by α. Our objective is to search for the values of $\lambda > 0$ such that there exists a α so that the minimum of f_1 and f_2 is larger than g. Observe that f_1 intersects the y-axis at $(0, 2\alpha - \alpha^2)$, which is below the point $(0, 5/4)$ where f_2 intersects the y-axis, because $2\alpha - \alpha^2 \leq 1\ \forall\alpha > 0$. On the other hand, f_1 intersects the x-axis at $((2-\alpha)/2, 0)$, which is to the left of $(5/4, 0)$ where f_2 intersects the x-axis, because $(2-\alpha)/2 < 1\ \forall\alpha > 0$. Therefore, we can conclude that RHS $= \min\{f_1, f_2\} = f_1$ due to these two facts, and the fact that f_1 is linear in λ. These arguments are geometrically illustrated in Fig. 1, where the family of f_1 parameterized by $\alpha > 0$ (in black) is always less than f_2 (in red). Hence, the maximum value of λ such that there exists α such that $g \leq f_1$ can be obtained by finding the value of λ where f_1 and g intersect. Equating f_1 and g yields

$$\lambda = \frac{2\alpha - \alpha^2}{4 + 2\alpha}, \text{ and } \lambda_{\max} = 3 - 2\sqrt{2}. \tag{3.47}$$

Consequently, there always exists a $\alpha > 0$ such that (3.46) is satisfied when $0 < \lambda < 3 - 2\sqrt{2}$.

Combining Case 1 and 2, we conclude that there always exists a $\alpha > 0$ such that (3.46) is satisfied provided that $\lambda < 3 - 2\sqrt{2}$. Under this condition, (3.44) gives us

$$V(t) \leq V_0 e^{-\beta t}, \tag{3.48}$$

where V_0 is the initial condition of $V(t)$. With this, we can also conclude from (3.39) that

$$\|\tilde{w}\|, \|\tilde{w}_x\| \to 0, \text{ as } t \to \infty. \tag{3.49}$$

Applying Agmon's ([2], Lemma 2.4) and Young's Inequality yields

$$\max_{x\in[0,1]} |\tilde{w}|^2 \leq \tilde{w}^2(0) + 2\|\tilde{w}\|\|\tilde{w}_x\| \leq \|\tilde{w}\|^2 + \|\tilde{w}_x\|^2, \tag{3.50}$$

where (3.31) is used, and we have thus proved that

$$\tilde{w}(x,t) \to 0 \quad \forall\ x \in [0,1], \quad \text{ as } t \to \infty. \tag{3.51}$$

According to the Fundamental Theorem of Calculus, triangle inequality, and Cauchy-Schwarz Inequality:

$$\begin{aligned} \tilde{w}_x(0,t) &= \tilde{w}_x(1,t) - \int_0^1 \tilde{w}_{xx} \mathrm{d}x \\ &\leq \alpha\left|\tilde{w}(1,t)\right| + \left(\int_0^1 \tilde{w}_{xx}^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \alpha\left|\tilde{w}(1,t)\right| + \|\tilde{w}_{xx}(\cdot,0)\| L e^{-\omega t}, \end{aligned} \tag{3.52}$$

where Lemma 3.2 has been imposed in the last inequality. As $t \to \infty$, $\tilde{w}(1,t) \to 0$ according to (3.51), and it can be concluded that $\tilde{w}_x(0,t) \to 0$ as $t \to \infty$. This concludes the proof for Lemma 3.3.

With Lemma 3.3, we are now positioned to present and prove the main result of the disturbance estimator.

THEOREM 3.1. *For any initial data $\tilde{d}_0$ which is finite, and $\lambda < 3 - 2\sqrt{2}$, there exists a constant $\alpha > 0$ such that the error for the disturbance estimation $\tilde{d}(t)$ converges to zero asymptotically,.*

Proof. According to (3.33),

$$\tilde{d} = d - \hat{d} = \tilde{u}_x(0,t) - \eta_x(0,t) = \tilde{w}_x(0,t), \tag{3.53}$$

which according to Lemma 3.3, is asymptotically stable.

REMARK 3.2. *The state in the system (3.20)-(3.22) is bounded in the sense of L^2 norm, as follows,*

$$\lim_{t\to\infty} \|\eta(\cdot,t)\| < \infty. \tag{3.54}$$

This can be verified by using the Lyapunov functional $W = \frac{1}{2}\|\eta(\cdot,t)\|^2$. Since η is bounded and $\tilde{w}$ is asymptotically stable in the sense of L^2 norm, y is bounded in the sense of L^2 norm. In addition, as u is bounded according to (2.19), ζ is also bounded in the sense of L^2 norm. We have thus proved that all the states in disturbance estimator stay bounded in the sense of L^2 norm.

REMARK 3.3. *The sufficient condition on λ for the asymptotic convergence of the disturbance estimator is conservative, since the majorization of $\dot{V}(t)$ in (3.42) using Poincaré Inequality is not tight.*

4 State Estimator Design

This section presents a state estimator utilizing the asymptotically convergent disturbance estimation signal. The state estimator is designed by using a copy of the plant model (2.7)-(2.10) with an error injection, i.e.

$$\hat{u}(x,t) = \hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t) + k(x)\tilde{u}(1,t), \tag{4.55}$$

$$\hat{u}_x(0,t) = qu(0,t) + \hat{d}(t), \tag{4.56}$$

$$\hat{u}_x(1,t) = U(t) + k_1\tilde{u}(1,t), \tag{4.57}$$

$$\hat{u}(x,0) = \hat{u}_0(x), \tag{4.58}$$

where $\hat{u}(x,t)$ represents the estimation of $u(x,t)$, and $k(x)$ and k_1 are, respectively, spatially-distributed and constant observer gains to be determined to achieve stability of state estimation error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$. Note that the disturbance estimation $\hat{d}(t)$ is injected into the boundary of the state estimator. The disturbance estimator is autonomous and upstream from the state estimator, so they are convergent independently.

Subtracting (4.55)-(4.58) from (2.7)-(2.10) yields the state estimation error dynamics:

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \lambda\tilde{u}(x,t) - k(x)\tilde{u}(1,t), \tag{4.59}$$

$$\tilde{u}_x(0,t) = \tilde{d}(t), \tag{4.60}$$

$$\tilde{u}_x(1,t) = -k_1\tilde{u}(1,t), \tag{4.61}$$

$$\tilde{u}(x,0) = u_0(x) - \hat{u}_0(x). \tag{4.62}$$

As $t \to \infty$, the disturbance estimation error $\tilde{d}(t)$ at $x = 0$ boundary vanishes when $\lambda < 3 - 2\sqrt{2}$, according to Theorem 3.1. Hence, when $\hat{d}(t)$ converges to $d(t)$, we recover a boundary condition with left end insulated, i.e.

$$\tilde{u}_x(0,t) = 0. \tag{4.63}$$

To determine the observer gains, we adopt the backstepping approach [2]. We seek a linear Volterra transformation that transforms the state of the error system $\tilde{u}(x,t)$ to the target state $\tilde{v}(x,t)$, by making use of the following expression:

$$\tilde{u}(x,t) = \tilde{v}(x,t) - \int_x^1 \ell(x,y)\tilde{v}(y,t)\mathrm{d}y, \tag{4.64}$$

which maps the error system (4.59), (4.61)-(4.63) to the exponentially stable heat equation (target system):

$$\tilde{v}(x,t) = \tilde{v}_{xx}(x,t), \tag{4.65}$$

$$\tilde{v}_x(0,t) = 0, \tag{4.66}$$

$$\tilde{v}_x(1,t) = 0, \tag{4.67}$$

where $\ell(x,y)$ is the gain kernel. To explicitly determine $\ell(x,y)$, we differentiate the transformation (4.64) with

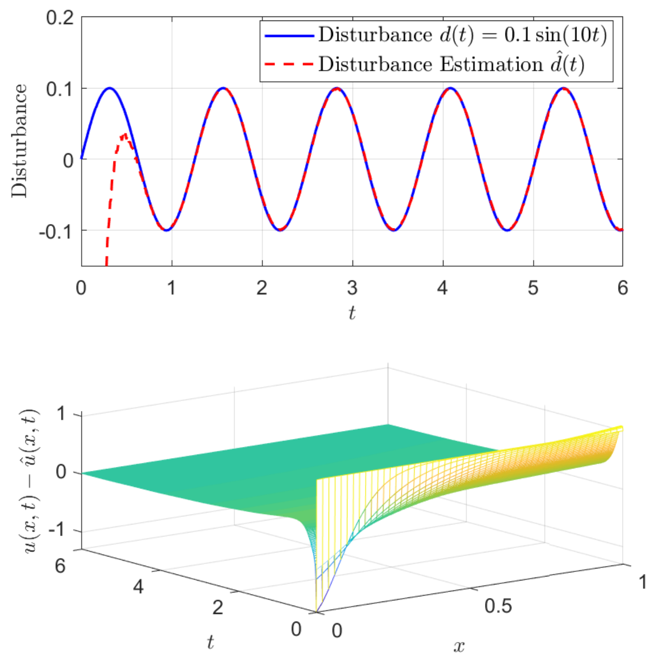

Figure 2: The convergence of disturbance estimator and backstepping state estimator.

respect to x and t, and conclude that $\ell(x,y)$ must satisfy the following Klein-Gordon PDE:

$$\ell_{xx}(x,y) - \ell_{yy}(x,y) = -\lambda\ell(x,y), \tag{4.68}$$

$$\ell_x(0,y) = 0, \tag{4.69}$$

$$\ell(x,x) = -\frac{\lambda}{2}x, \tag{4.70}$$

in which the boundary condition (4.69)-(4.70) emerges from evaluating (4.64) together with the boundary conditions (4.63) and (4.61). An unique and closed-form analytic solution exists for the kernel $p(x,y)$ [2]:

$$\ell(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(y^2 - x^2)}\right)}{\sqrt{\lambda(y^2 - x^2)}}, \tag{4.71}$$

where $I_1(\cdot)$ is the Modified Bessel Function of the first kind. Moreover, the observer gains are computed as

$$k(x) = -\ell_y(x,1), \quad k_1 = -\ell(1,1). \tag{4.72}$$

Therefore, the observer gains can be determined offline using the kernel PDE solution (4.71). It can also be proven that the linear Volterra transformation (4.64) is invertible [2]. Thus, the exponential stability of the target system (4.65)-(4.67) implies the stability of the original error system (4.59), (4.61)-(4.63).

5 Numerical Simulation and Discussion

In this section, we demonstrate the effectiveness of the proposed estimators. The plant model (2.7)-(2.10), disturbance estimator (3.20)-(3.26), and the backstepping state estimator (4.55)-(4.58) are implemented in MATLAB. The finite difference method is employed in spatial discretization. 51 points has been utilized to discretize in space, and the spatial discretization step is $dx = 1/50$. The simulation end time is chosen as $T = 6$s. We use reaction coefficient $\lambda = -1$, constant $q = 0.5$, the disturbance $d(t) = 0.1\sin(5t)$, and input $U(t) = 0$ for an illustrative example. As demonstrated in Figure 2, with an appropriate selection of design variable $\alpha > 0$, the disturbance estimation $\hat{d}(t)$ converges to its true value $d(t)$ asymptotically, and the backstepping observer reconstructs the actual state asymptotically.

6 Conclusion

In this paper, we propose and rigorously analyze a combined disturbance and state estimator for a class of unstable reaction-advection-diffusion PDEs, subject to unknown boundary disturbance. A sufficient condition on the reaction coefficient is derived, for which the disturbance estimation error is asymptotically stable. The disturbance estimate is combined with a backstepping state observer to also yield asymptotically convergent state estimates. The convergence of the estimators are analyzed by Lyapunov stability analysis. The results of this paper can be applied to ADRC where the disturbance estimate is required to attenuate the actual disturbance in a feedback controller design, as the plant model becomes unstable for certain combination of λ and q. Future work will also examine the necessary condition on reaction coefficient λ for the disturbance estimator to be asymptotically convergent.

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References

[1] Griffiths, David F., Dold, John W., Silvester, David J. "Essential Partial Differential Equations Analytical and Computational Aspects." Switzerland: Springer International Publishing, 2015.

[2] Krstic, Miroslav, and Smyshlyaev, Andrey. "Boundary control of PDEs: A course on backstepping designs." Vol. 16. Siam, 2008.

[3] Feng, Hongyinping, and Guo, Bao-Zhu. "New unknown input observer and output feedback stabilization for uncertain heat equation." Automatica 86 (2017): 1-10.

[4] Vazquez, Rafael, and Krstic, Miroslav. "Explicit output feedback stabilization of a thermal convection loop by continuous backstepping and singular perturbations." American Control Conference, 2007. ACC'07. IEEE, 2007.

[5] Doyle, Marc, Fuller, Thomas F., and Newman, John. "Modeling of galvanostatic charge and discharge of the lithium/polymer/insertion cell." Journal of the Electrochemical Society 140.6 (1993): 1526-1533.

[6] Avalos, George. "The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustics." Abstract and Applied Analysis. Vol. 1. No. 2. Hindawi, 1996.

[7] Liu, Jun-Jun, and Wang, Jun-Min. "Active disturbance rejection control and sliding mode control of one-dimensional unstable heat equation with boundary uncertainties." IMA Journal of Mathematical Control and Information 32.1 (2013): 97-117.

[8] Ahmed-Ali, Tarek, Giri, Fouad, Krstic, Miroslav, Burlion, Laurent, and Lamnabhi-Lagarrigue, Françoise. "Adaptive boundary observer for parabolic PDEs subject to domain and boundary parameter uncertainties." Automatica 72 (2016): 115-122.

[9] Pisano, Alessandro, and Orlov, Yury. "Boundary second-order sliding-mode control of an uncertain heat process with unbounded matched perturbation." Automatica 48.8 (2012): 1768-1775.

[10] Wang, Ji, et al. "Exponential regulation of the anti-collocatedly disturbed cage in a wave PDE-modeled ascending cable elevator." Automatica 95 (2018): 122-136.

[11] Guo, Bao-Zhu, and Jin, Feng-Fei. "Output feedback stabilization for one-dimensional wave equation subject to boundary disturbance." IEEE Transactions on Automatic control 60.3 (2015): 824-830.

[12] Han, Jingqing. "From PID to active disturbance rejection control." IEEE transactions on Industrial Electronics 56.3 (2009): 900-906.

[13] Guo, Bao-Zhu, and Zhao, Zhi-liang. "On the convergence of an extended state observer for nonlinear systems with uncertainty." Systems & Control Letters 60.6 (2011): 420-430.

[14] Wang, Ji, et al. "Disturbance Estimation of a Wave PDE on a Time-varying Domain." 2017 Proceedings of the Conference on Control and its Applications. Society for Industrial and Applied Mathematics, 2017.

[15] Feng, Hongyinping, and Guo, Bao-Zhu. "Output feedback stabilization of an unstable wave equation with general corrupted boundary observation." Automatica 50.12 (2014): 3164-3172.

[16] Tang, Shuxia, and Xie, Chengkang. "State and output feedback boundary control for a coupled PDEODE system." Systems & Control Letters 60.8 (2011): 540-545.

Appendices

A Well-Posedness for Linear BVP

The following lemma is well-established to deal with the well-posedness of a linear BVP [1]. The proof for the lemma is omitted here.

LEMMA A.1. *Suppose the BVP under consideration is written in the form*

$$(\mathcal{H}u)(x,t) = \mathcal{F}(x,t), \tag{A.1}$$

where $\mathcal{H}$ contains both the differential and boundary operators, and $\mathcal{F}$ the data terms comprising the right hand side of the differential equation and the boundary conditions. Moreover, suppose

1. *$\mathcal{H}$ is linear.*
2. *$\mathcal{H}$ is inverse monotone: $\mathcal{H}v \geq 0$ implies $v \geq 0$.*
3. *A bounded and non-negative comparison function $\phi(x)$ exists, such that $\mathcal{H}\phi(x) \geq 1$ for all $x \in [0,1]$.*

If an appropriate norm $\|\cdot\|_u$ is defined such that

$$-\|\mathcal{F}\|_u \leq \mathcal{F} \leq \|\mathcal{F}\|_u, \tag{A.2}$$

then the problem $(\mathcal{H}u) = \mathcal{F}$ is well-posed:

$$-\|\mathcal{F}\|_u \phi \leq u \leq \|\mathcal{F}\|_u \phi \tag{A.3}$$

at all points $x \in [0,1]$, which means that

$$\max_{x\in[0,1]} |u| \leq \gamma \|\mathcal{F}\|_u, \tag{A.4}$$

where $\gamma = \max_{x\in[0,1]} \phi$.