Stabilization for a class of delayed coupled PDE-ODE systems with boundary control

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***Abstract*— A class of coupled PDE-ODE systems with delays at the input and two-directional interconnection at the interfaces (one single point) is discussed in this paper. Exponential stability in the sense of the corresponding norms for the original closed-loop system with the derived controller is obtained by finally transforming the system into an exponentially stable PDEs-ODE cascade with a boundary feedback backstepping controller, and the result is also rigidly proved.**

***Index Terms*— Coupled system; Input delay; Boundary control; Backstepping**

I. INTRODUCTION

Problems concerning coupled systems and time delay systems have been interesting areas for long, both exist in many practical control systems such as electromagnetic coupling, mechanical coupling, and coupled chemical reactions, and researchers have worked out fruitful results in both areas.

Controllability of coupled PDE-PDE systems has been successfully studied in [13], [14], [15]. Boundary feedback control problem of cascaded PDE-ODE systems, where the interconnection between PDE and ODE is one-directional, has been beautifully solved in [4], [5], [7], [8], [9] with backstepping method. Designing the boundary feedback controllers for coupled PDE-PDE systems as well as coupled PDE-ODE systems, e.g. [10], [11], [12], is an interesting area just opening up for research.

For ODE systems with time delays, there already exist fruitful results, e.g., [3], [7]. Control of PDE systems with time delays is an inspiring area where there are a few established results, e.g., [1], [6], [7].

The class of systems under consideration in this paper is strongly coupled PDE-ODE systems with input delays. Here the interconnection between PDE and ODE is at one point and two-directional, that is, the ODE acts back on PDE at the same time as the PDE acts on the ODE. The control problem for this system is motivated by applications in chemical process control, heat diffusion control, combustion, and other areas.

This paper is organized as follows. In Section II, the problem is stated. The method of establishing an interconnection to transfer the input time delay into a space extension, first introduced by Miroslav Krstic, is employed to transform the system into a cascade of a PDE with a coupled PDE-ODE system without time delay. Backstepping method is also introduced. In Section III, a boundary feedback controller is designed to stabilize the coupled PDE-ODE system under consideration. In Section IV, exponential stability of the resulted closed-loop system with the control law is analyzed. In Section V, some comments are made on boundary feedback control of the coupled PDE-ODE systems.

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II. PROBLEM STATEMENT

In this paper we consider the following coupled PDE-ODE systems,

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{1}$$
$$u_t(x,t) = u_{xx}(x,t), x \in (0,l), \tag{2}$$
$$u(0,t) = CX(t), \tag{3}$$
$$u(l,t) = U(t-d), \tag{4}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, the pair (A,B) is assumed to be controllable; $u(x,t) \in \mathbb{R}$ is the PDE state, C^T is a constant vector; $U(t)$ is the scalar input, and d is the time delay, which is known constant. The coupled systems are depicted in Figure 1. The control objective is to exponentially stabilize the system.

Firstly, it can be easily verified that the system $(1)-(4)$ can also be represented by the following system

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{5}$$
$$u_t(x,t) = u_{xx}(x,t),\ x \in (0,l), \tag{6}$$
$$u(0,t) = CX(t), \tag{7}$$
$$u(l,t) = v(l,t), \tag{8}$$
$$v_t(x,t) = v_x(x,t),\ x \in [l, l+d), \tag{9}$$
$$v(l+d,t) = U(t). \tag{10}$$

Moreover, the state of the input delay dynamics is known explicitly since it holds that

$$v(x,t) = U(t+x-l-d), \quad x \in [l, l+d].$$

Then, backstepping method is to be employed, which is to seek an invertible transformation $(X,u,v) \mapsto (X,w,z)$ to

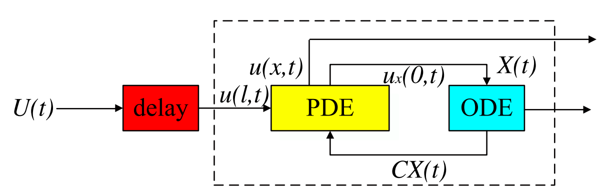

Fig. 1. The coupled PDE-ODE system with input time delay

transform the system $(5)-(10)$ into an exponentially stable target system, e.g., the following system

$$\dot{X}(t) = (A + BK)X(t) + Bw_x(0,t) \quad (11)$$
$$w_t(x,t) = w_{xx}(x,t),\ x \in (0,l) \quad (12)$$
$$w(0,t) = 0 \quad (13)$$
$$w(l,t) = z(l,t) \quad (14)$$
$$z_t(x,t) = z_x(x,t),\ x \in [l, l+d) \quad (15)$$
$$z(l+d,t) = 0 \quad (16)$$

is an exponentially stable PDE-ODE cascade where K is chosen such that $A + BK$ is Hurwitz, then exponential stabilization of the original system can be achieved with the invertibility of the transformation.

III. CONTROLLER DESIGN

Considering a backstepping transformation of the form

$$X(t) = X(t), \quad (17)$$
$$w(x,t) = u(x,t) - \int_0^x \kappa(x,y)u(y,t)dy - \Phi(x)X(t),\ x \in [0,l], \quad (18)$$
$$z(x,t) = v(x,t) - \int_l^x p(x-y)v(y,t)dy - \int_0^l \theta(x,y)u(y,t)dy - \Delta(x)X(t),\ x \in [l,l+d], \quad (19)$$

where the kernel functions κ, Φ, p, θ and Δ are to be determined later.

With a lengthy calculation, a sufficient condition for $(11)-(16)$ to hold is that the kernel functions satisfy some governing equations.

Firstly, the kernel functions $\kappa(x,y)$ and $\Phi(x)$ satisfy

$$\kappa_{xx}(x,y) = \kappa_{yy}(x,y),\ 0 \le y \le x \le l, \quad (20)$$
$$\frac{d}{dx}\kappa(x,x) = 0, \quad (21)$$
$$\kappa(x,0) = \Phi(x)B \quad (22)$$

and

$$\Phi''(x) - \Phi(x)A - \kappa_y(x,0)C = 0,\ 0 \le y \le x \le l, \quad (23)$$
$$\Phi(0) = C, \quad (24)$$
$$\Phi'(0) = K - \kappa(0,0)C. \quad (25)$$

The solution can be found explicitly in [12] as

$$\Phi(x) = \begin{pmatrix} C & K - CBC \end{pmatrix} e^{Dx} \begin{pmatrix} I \\ 0 \end{pmatrix},$$
$$\kappa(x,y) = \Phi(x-y)B = \begin{pmatrix} C & K - CBC \end{pmatrix} e^{D(x-y)} \begin{pmatrix} I \\ 0 \end{pmatrix} B,$$

where the matrix I stands for the unit matrix, and

$$D = \begin{pmatrix} 0 & A \\ I & -BC \end{pmatrix}.$$

Secondly, the kernel functions $\theta(x,y)$ and $\Delta(x)$ satisfy

$$\theta_x(x,y) = \theta_{yy}(x,y),\ (x,y) \in [l, l+d] \times (0,l), \quad (26)$$
$$\theta(l,y) = \kappa(l,y), \quad (27)$$
$$\theta(x,0) = \Delta(x)B, \quad (28)$$
$$\theta(x,l) = 0 \quad (29)$$

and

$$\Delta'(x) - \Delta(x)A - \theta_y(x,0)C = 0,\ (x,y) \in [l, l+d] \times (0,l), \quad (30)$$

where

$$\theta(l,y) = \kappa(l,y), \Delta(l) = \Phi(l)$$

Through the method of successive approximations and some other techniques, it can be shown that there exist unique classic solutions for these equations.

Lastly, the kernel function $p(x-y)$ satisfies

$$p(x-l) = -\theta_y(x,l),\ l \le y \le x \le l+d, \quad (31)$$

that is

$$p(s) = -\theta_y(l+s,l), \quad s \in [0,d].$$

The backstepping transformation $(X,u,v) \mapsto (X,w,z)$ $(17)-(19)$ is invertible, and the inverse transformation is postulated in the following form

$$X(t) = X(t), \quad (32)$$
$$u(x,t) = w(x,t) + \int_0^x \iota(x,y)w(y,t)dy + \Psi(x)X(t), x \in [0,l], \quad (33)$$
$$v(x,t) = z(x,t) + \int_l^x q(x-y)z(y,t)dy + \int_0^l \vartheta(x,y)w(y,t)dy + \Upsilon(x)X(t), x \in [l, l+d], \quad (34)$$

where the kernel functions ι, Ψ, q, ϑ and Υ can be determined as follows:

$$\Psi(x) = \begin{pmatrix} C & K \end{pmatrix} e^{Ex} \begin{pmatrix} I \\ 0 \end{pmatrix},$$
$$\iota(x,y) = \Psi(x-y)B = \begin{pmatrix} C & K \end{pmatrix} e^{E(x-y)} \begin{pmatrix} I \\ 0 \end{pmatrix} B,$$

where

$$E = \begin{pmatrix} 0 & A+BK \\ I & 0 \end{pmatrix},$$

$$\Upsilon(x) = \begin{pmatrix} C & K \end{pmatrix} e^{El} \begin{pmatrix} I \\ 0 \end{pmatrix} e^{(A+BK)(x-l)},$$

$$\vartheta(x,y) = \sum_{n=1}^{\infty} D_n(x) \sin\frac{n\pi y}{l} + \Upsilon(x)B\frac{l-y}{l},$$

where

$$D_n(x) = D_n(l)e^{-\frac{n^2\pi^2}{l^2}(x-l)} + \frac{2}{n\pi}e^{-\frac{n^2\pi^2}{l^2}x}\int_x^l e^{\frac{n^2\pi^2}{l^2}\eta}\Upsilon'(\eta)d\eta B$$

and

$$D_n(l) = \frac{2}{l}\int_0^l \left(\vartheta(l,\xi) - \Upsilon(l)B\frac{l-\xi}{l}\right)\sin\frac{n\pi\xi}{l}d\xi,$$

$$q(s) = -\vartheta_y(l+s,l), \quad s\in[0,d].$$

Let $x = l+d$ in (19) and use the boundary conditions (10) and (16), then the controller is obtained as follows

$$U(t) = \int_0^l \theta(l+d,y)u(y,t)dy + \int_l^{l+d} p(l+d-y)v(y,t)dy + \Delta(l+d)X(t). \tag{35}$$

IV. STABILITY ANALYSIS

In order to establish the exponential stability result in appropriate norm for system (11) – (16), the following lemmas are to be proved firstly.

Lemma 1: Considering the change of variable

$$\varpi(x,t) = w(x,t) - \frac{x}{l}z(l,t)$$

and the resulting cascade system of PDEs

$$\dot{X}(t) = (A+BK)X(t) + B\varpi_x(0,t) + B\frac{z(l,t)}{l}, \tag{36}$$

$$\varpi_t(x,y) = \varpi_{xx}(x,t) - \frac{x}{l}z_t(l,t),\ x\in(0,l), \tag{37}$$

$$\varpi(0,t) = 0, \tag{38}$$

$$\varpi(l,t) = 0, \tag{39}$$

$$z_t(x,t) = z_x(x,t),\ x\in[l,l+d), \tag{40}$$

$$z(l+d,t) = 0, \tag{41}$$

we can obtain that

$$\Xi(t) \le \Xi(0)e^{-C_1 t},\ \forall t\ge 0,$$

where

$$\Xi(t) = X^TPX + \frac{4|PB|^2}{\lambda_{\min}(Q)}\|\varpi(t)\|^2_{L^2(0,l)} + \frac{4|PB|^2}{\lambda_{\min}(Q)l^2}\int_l^{l+d} e^{x-l}z^2(x,t)dx + \frac{2|PB|^2l^3}{3\lambda_{\min}(Q)}\int_l^{l+d} e^{x-l}z_x^2(x,t)dx$$

and

$$C_1 = \min\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{\pi^2-8}{4l^2}, 1\}.$$

Here the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^TP = -Q$$

for some $Q = Q^T > 0$. For simplicity, in the sequel, the symbol $\|\cdot\|$ denotes the L^2 norm.

Proof: Based on the Wirtinger inquality [2] (a tight version of Poincare's inequality) and Young's inequality , it can be obtained that

$$\frac{d}{dt}X^TPX \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 + \frac{4|PB|^2}{\lambda_{\min}(Q)}\|\varpi_x\|^2 + \frac{4|PB|^2}{\lambda_{\min}(Q)}\frac{z^2(l,t)}{l^2}, \tag{42}$$

$$\frac{d}{dt}\frac{1}{2}\|\varpi(t)\|^2 = -\frac{1}{2}\|\varpi_x(t)\|^2 - \frac{1}{l^2}(\frac{\pi^2}{8}-1)\|\varpi(t)\|^2 + \frac{l^3}{12}z_x^2(l,t), \tag{43}$$

$$\frac{d}{dt}\int_l^{l+d} e^{x-l}z^2(x,t)dx = -z^2(l,t) - \int_l^{l+d} e^{x-l}z^2(x,t)dx, \tag{44}$$

$$\frac{d}{dt}\int_l^{l+d} e^{x-l}z_x^2(x,t)dx = -z_x^2(l,t) - \int_l^{l+d} e^{x-l}z_x^2(x,t)dx. \tag{45}$$

Combining (42)-(45), the result of this lemma can be obtained. ■

In the next two lemmas, the Lyapunov function $\Xi(t)$ is to be related with the norm

$$\Pi(t) := |X(t)|^2 + \|w(\cdot,t)\|^2 + \|z(\cdot,t)\|^2_{H^1(l,l+d)}$$

to derive exponential stability of the transformed system (11) – (16) in the sense of the norm $\Pi(t)$.

Lemma 2:

$$\Xi(t) \le C_2\Pi(t), \tag{46}$$

where

$$C_2 = \max\{\lambda_{\max}(P), \frac{8|PB|^2}{\lambda_{\min}(Q)}, \frac{4|PB|^2}{\lambda_{\min}(Q)l^2}e^d, \frac{2(l^2+4)l|PB|^2}{3\lambda_{\min}(Q)}e^d\}.$$

Proof: Using the Cauchy-Schwartz inequality, it can be obtained that

$$\|\varpi(t)\|^2 \le 2\|w(t)\|^2 + \frac{2l}{3}z^2(l,t).$$

Thus,

$$\Xi(t) \le X^TPX + \frac{4|PB|^2}{\lambda_{\min}(Q)}\left(2\|w(t)\|^2 + \frac{2l}{3}z^2(l,t)\right) + \frac{4|PB|^2}{\lambda_{\min}(Q)l^2}\int_l^{l+d} e^dz^2(x,t)dx + \frac{2|PB|^2l^3}{3\lambda_{\min}(Q)}\int_l^{l+d} e^dz_x^2(x,t)dx,$$

which completes the proof. ■

Lemma 3:

$$\Pi(t) \le C_3\Xi(t), \tag{47}$$

where

$$C_3=\max\{\frac{1}{\lambda_{\max}(P)},\frac{\max\{2,\frac{8ld}{3}\}\lambda_{\min}(Q)}{4|PB|^2},\frac{l^2\lambda_{\min}(Q)}{4|PB|^2},\frac{3\left(\max\{2,\frac{8ld}{3}\}+1\right)\lambda_{\min}(Q)}{2|PB|^2l^3}\}.$$

Proof: First, using the Cauchy-Schwartz inequality, Young's and Agmon's inequalities, it can be obtained that

$$\|w(t)\|^2\le\max\{2,\frac{8ld}{3}\}\left(\|\varpi(t)\|^2+\|z_x(t)\|^2\right).$$

Then,

$$\begin{aligned}\Pi(t)\le&\frac{1}{\lambda_{\max}(P)}X^TPX\\&+\max\{2,\frac{8ld}{3}\}\left(\|\varpi(t)\|^2+\|z_x(t)\|^2\right)\\&+\int_l^{l+d}e^{x-l}\left(z^2(x,t)+z_x^2(x,t)\right)dx,\end{aligned}$$

from which the proof is completed. ■

From the results above, exponential stability of the transformed system (11) − (16) in the sense to the norm $\Pi(t)$ can be derived.

Lemma 4:

$$\Pi(t)\le C_2C_3\Pi(0)e^{-C_1t},\ \forall t\ge0. \tag{48}$$

Then, the norm $\Pi(t)$ is to be related with the norm

$$\Omega(t):=|X(t)|^2+\|u(\cdot,t)\|^2+\|v(\cdot,t)\|^2_{H^1(l,l+d)}$$

to derive exponential stability of the original system (5) − (10) in the sense of the norm $\Omega(t)$.

Lemma 5:

$$\Pi(t)\ \le\ C_4\Omega(t), \tag{49}$$
$$\Omega(t)\ \le\ C_5\Pi(t), \tag{50}$$

where

$$\begin{aligned}C_4=&3\|\Phi\|^2+4\|\Omega\|^2+5\|\Omega'\|^2+3l\|\Phi B\|^2\\&+4\int_l^{l+d}\int_0^l\theta^2(x,y)dydx\\&+5\int_l^{l+d}\int_0^l\theta_x^2(x,y)dydx+4d\int_l^{l+d}\theta_y^2(x,l)dx\\&+5\left(\theta_y^2(l,l)+d\int_l^{l+d}\theta_{xy}^2(x,l)dx\right)+13,\\C_5=&3\|\Psi\|^2+4\|\Upsilon\|^2+5\|\Upsilon'\|^2+3l\|\Psi B\|^2\\&+4\int_l^{l+d}\int_0^l\vartheta^2(x,y)dydx\\&+5\int_l^{l+d}\int_0^l\vartheta_x^2(x,y)dydx+4d\int_l^{l+d}\vartheta_y^2(x,l)dx\\&+5\left(\vartheta_y^2(l,l)+d\int_l^{l+d}\vartheta_{xy}^2(x,l)dx\right)+13.\end{aligned}$$

Proof: Since

$$\begin{aligned}\|w(t)\|^2\le&3\left(1+l\|\Phi B\|^2\right)\|u(t)\|^2+3\|\Phi\|^2|X|^2,\\\|z(t)\|^2\le&4\left(1+\int_l^{l+d}\int_0^{x-l}\theta_y^2(l+s,l)dsdx\right)\|v(t)\|^2\\&+4\left(\int_l^{l+d}\int_0^l\theta^2(x,y)dydx\right)\|u(t)\|^2\\&+4\|\Omega\|^2|X|^2,\\\|z_x(t)\|^2\le&5\|v_x(t)\|^2+5\left(\theta_y^2(l,l)\right.\\&\left.+\int_l^{l+d}\int_0^{x-l}\theta_{xy}^2(l+s,l)dsdx\right)\|v(t)\|^2\\&+5\left(\int_l^{l+d}\int_0^l\theta_x^2(x,y)dydx\right)\|u(t)\|^2\\&+5\|\Omega'\|^2|X|^2,\end{aligned}$$

the inequality (49) is obtained. In a similar manner the inequality (50) can also be obtained. ■

It can be derived that the constants f and g are finite. Then, it holds that

$$\Omega(t)\le C_2C_3C_4C_5\Omega(0)e^{-C_1t},\ \forall t\ge0.$$

Thus, exponential stability of the original closed system with the control law can be derived. In sum,we obtain the following main theorem.

Theorem 1: The closed-loop system consisting of the plant (1) − (4) and the control law (35) has a solution $(X(\cdot),u(\cdot,\cdot),v(\cdot,\cdot))\in\mathbb{R}^n\times C\left([0,\infty],L^2[0,l]\times H^1[l,l+d]\right)$ for initial state $(X(0),u(\cdot,0),v(\cdot,0))\in\mathbb{R}^n\times L^2[0,l]\times H^1[l,l+d]$, and the state $(X(t),u(\cdot,t),v(\cdot,t))$ is exponentially stabilized in the sense of the norm $\Omega(t)$.

V. COMMENTS

Boundary feedback stabilization for coupled PDE-ODE control systems with time delay is an original area with many problems to be considered. More general and complicated systems, such as

$$\begin{aligned}\dot X(t)=&AX(t)+Bu_x(0,t),\\u_t(x,t)=&u_{xx}(x,t)+b(x)u_x(x,t)\\&+c(x)u(x,t)+\int_0^xd(x,y)u(y,t)dy,x\in(0,l),\\u_x(0,t)=&-qu(0,t)+CX(t),\\u(l,t)=&U(t-d),\end{aligned}$$

where $b(x),c(x),d(x,y)$ are arbitrary continuous functions, are being worked on.

More interesting areas, such as boundary feedback stabilization of coupled PDE-PDE systems with time delays, are also subjects of the ongoing research.

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