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Sliding Mode Control to the Stabilization of a Linear 2×2 Hyperbolic System with Boundary Input Disturbance

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Abstract— **In this paper, sliding mode control approach is used to stabilize a 2×2 system of first-order linear hyperbolic PDEs subject to boundary input disturbance. Disturbance rejection is achieved, and with the designed first-order sliding mode controller, the resulting closed-loop system admits a unique (mild) solution without chattering. Convergence to the chosen infinite-dimensional sliding surface of state trajectories takes place in a finite time. Then on the sliding surface, the system is exponentially stable with a decay rate depending on the spatially varying system coefficients. A simulation example is presented to illustrate the effectiveness and performance of sliding mode control method.**

Index Terms— **Linear 2×2 hyperbolic system; First-order sliding mode control; Disturbance rejection; Backstepping.**

I. Introduction

Linear 2×2 hyperbolic systems have wide physical backgrounds, such as oil wells [1], transmission lines [2], road traffic [3], open channels [4], and so on. Due to their practical and theoretical values, stabilization of these systems has been a topic of active research (see, e.g., [5], [6], [7]). Also, quasilinear 2×2 systems of hyperbolic PDEs have received some attention (see, e.g., [8], [9]).

System uncertainties and disturbances are common problems, which sometimes can worsen the system performance or even lead to instability, and thus need to be taken into account. Disturbance attenuation and disturbance rejection are desirable in system control design. There have been some research results utilizing different methods to deal with particular types of boundary input disturbances (see, e. g., [10], [11], [12]) in distributed parameter systems.

Sliding mode control technique has been studied for decades and is characterized by its high simplicity and robustness among the existing methods. Recently, this approach has been generalized to distributed parameter systems. For example, it is used to reject the more general boundary input disturbance in wave equation [13], Euler-Bernoulli beam equation [14] and Schrödinger equation [15].

The system considered in this paper is a 2×2 system of first-order linear hyperbolic PDEs with spatially varying coefficients and a boundary input disturbance. The control objective is to stabilize the system while rejecting the disturbance. The control method employed is sliding mode boundary control. Since the designed controller is first-order and thus continuous (see, e. g. [16], [17], [18], [19]), chattering is avoided in the resulting closed-loop system.

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II. Problem Statement

In this paper, we intend to employ sliding mode control to stabilize the following system (see, Fig. 1):

$$u_t(x,t) = -\varepsilon_1(x)u_x(x,t) + c_1(x)v(x,t) \tag{1}$$

$$v_t(x,t) = \varepsilon_2(x)v_x(x,t) + c_2(x)u(x,t) \tag{2}$$

$$u(0,t) = qv(0,t) \tag{3}$$

$$v(1,t) = U(t) + d(t), \tag{4}$$

where $u(x,t), v(x,t)$ are system states with $x \in [0,1]$, $t > 0$; $U(t)$ is control input; $d(t)$ is external disturbance at the control end.

Here are some assumptions:
1. $\varepsilon_1(x), \varepsilon_2(x) \in C^1[0,1]$, $\varepsilon_1(x), \varepsilon_2(x) > 0$,
2. $c_1(x), c_2(x) \in C[0,1]$,
3. $q \neq 0$,
4. $d(t)$ and $\dot{d}(t)$ are bounded measurable, that is, $|d(t)| \leq M$, $|\dot{d}(t)| \leq M$ for some $M > 0$ and all $t \geq 0$,
5. Initial data $u_0(x), v_0(x) \in L^2[0,1]$.

Following [7], we introduce a backstepping transformation

$$\alpha(x,t) = u(x,t) - \int_0^x K^{uu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)v(\xi,t)d\xi \tag{5}$$

$$\beta(x,t) = v(x,t) - \int_0^x K^{vu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)v(\xi,t)d\xi, \tag{6}$$

in which the continuous kernel functions are uniquely determined by the following system:

$$\varepsilon_1(x)K_x^{uu} + \varepsilon_1(\xi)K_\xi^{uu} = -\varepsilon_1'(\xi)K^{uu} - c_2(\xi)K^{uv} \tag{7}$$

$$\varepsilon_1(x)K_x^{uv} - \varepsilon_2(\xi)K_\xi^{uv} = \varepsilon_2'(\xi)K^{uv} - c_1(\xi)K^{uu} \tag{8}$$

$$\varepsilon_2(x)K_x^{vu} - \varepsilon_1(\xi)K_\xi^{vu} = \varepsilon_1'(\xi)K^{vu} + c_2(\xi)K^{vv} \tag{9}$$

$$\varepsilon_2(x)K_x^{vv} + \varepsilon_2(\xi)K_\xi^{vv} = -\varepsilon_2'(\xi)K^{vv} + c_1(\xi)K^{vu} \tag{10}$$

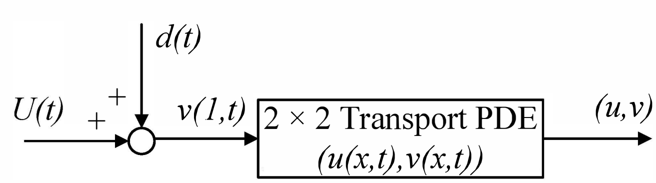


Fig. 1. Block diagram of the system (1) – (4)

with boundary conditions

$$K^{uu}(x,0)=\frac{\varepsilon_2(0)}{q\varepsilon_1(0)}K^{uv}(x,0),\ K^{uv}(x,x)=\frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{11}$$

$$K^{vu}(x,x)=-\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)},\ K^{vv}(x,0)=\frac{q\varepsilon_1(0)}{\varepsilon_2(0)}K^{vu}(x,0). \tag{12}$$

The transformation $(5)-(6)$ is invertible and the inverse is:

$$u(x,t)=\alpha(x,t)+\int_0^x L^{\alpha\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(\xi,t)d\xi \tag{13}$$

$$v(x,t)=\beta(x,t)+\int_0^x L^{\beta\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(\xi,t)d\xi, \tag{14}$$

where the continuous kernel functions are uniquely determined by the following system:

$$\varepsilon_1(x)L_x^{\alpha\alpha}+\varepsilon_1(\xi)L_\xi^{\alpha\alpha}=-\varepsilon_1'(\xi)L^{\alpha\alpha}+c_1(x)L^{\beta\alpha} \tag{15}$$

$$\varepsilon_1(x)L_x^{\alpha\beta}-\varepsilon_2(\xi)L_\xi^{\alpha\beta}=\varepsilon_2'(\xi)L^{\alpha\beta}+c_1(x)L^{\beta\beta} \tag{16}$$

$$\varepsilon_2(x)L_x^{\beta\alpha}-\varepsilon_1(\xi)L_\xi^{\beta\alpha}=\varepsilon_1'(\xi)L^{\beta\alpha}-c_2(x)L^{\alpha\alpha} \tag{17}$$

$$\varepsilon_2(x)L_x^{\beta\beta}+\varepsilon_2(\xi)L_\xi^{\beta\beta}=-\varepsilon_2'(\xi)L^{\beta\beta}-c_2(x)L^{\alpha\beta} \tag{18}$$

with boundary conditions

$$L^{\alpha\alpha}(x,0)=\frac{\varepsilon_2(0)}{q\varepsilon_1(0)}L^{\alpha\beta}(x,0),\ L^{\alpha\beta}(x,x)=\frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{19}$$

$$L^{\beta\alpha}(x,x)=-\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)},\ L^{\beta\beta}(x,0)=\frac{q\varepsilon_1(0)}{\varepsilon_2(0)}L^{\beta\alpha}(x,0). \tag{20}$$

The transformation $(5)-(6)$ maps the system $(1)-(4)$ into the following system (see, Fig. 2):

$$\alpha_t(x,t)=-\varepsilon_1(x)\alpha_x(x,t) \tag{21}$$

$$\beta_t(x,t)=\varepsilon_2(x)\beta_x(x,t) \tag{22}$$

$$\alpha(0,t)=q\beta(0,t) \tag{23}$$

$$\begin{aligned}\beta(1,t)=&U(t)+d(t)-\int_0^1\alpha(\xi,t)\Big(K^{vu}(1,\xi)\\&+\int_\xi^1 K^{vu}(1,\eta)L^{\alpha\alpha}(\eta,\xi)d\eta\\&+\int_\xi^1 K^{vv}(1,\eta)L^{\beta\alpha}(\eta,\xi)d\eta\Big)d\xi\\&-\int_0^1\beta(\xi,t)\Big(K^{vv}(1,\xi)\\&+\int_\xi^1 K^{vu}(1,\eta)L^{\alpha\beta}(\eta,\xi)d\eta\\&+\int_\xi^1 K^{vv}(1,\eta)L^{\beta\beta}(\eta,\xi)d\eta\Big)d\xi.\end{aligned} \tag{24}$$

III. Control design

Consider the systems $(1)-(4)$ and $(21)-(24)$ in the state Hilbert space $\mathbf{H}=\left(L^2(0,1)\right)^2$ with the norm induced by the following inner product

$$\begin{aligned}&<(f_1,\ g_1)^T,(f_2,\ g_2)^T>\\&=\int_0^1\left(\frac{2-x}{\varepsilon_1(x)}f_1(x)\overline{f_2(x)}+\frac{2q^2(1+x)}{\varepsilon_2(x)}g_1(x)\overline{g_2(x)}\right)dx,\\&\quad\forall\ (f_1,\ g_1)^T,(f_2,\ g_2)^T\in\mathbf{H}.\end{aligned} \tag{25}$$

A. Sliding surface

Define energy of the system $(21)-(24)$ by:

$$E(t)=\frac{1}{2}\int_0^1\left(\frac{2-x}{\varepsilon_1(x)}|\alpha(x,t)|^2+\frac{2q^2(1+x)}{\varepsilon_2(x)}|\beta(x,t)|^2\right)dx, \tag{26}$$

then

$$\begin{aligned}\dot{E}(t)=&-\frac{1}{2}|\alpha(1,t)|^2+2q^2|\beta(1,t)|^2\\&-\frac{1}{2}\int_0^1\left(|\alpha(x,t)|^2+2q^2|\beta(x,t)|^2\right)dx.\end{aligned} \tag{27}$$

Choose a sliding surface

$$S_{(\alpha,\beta)^T}(t)=\beta(1,t)=0, \tag{28}$$

i.e.,

$$S_{(\alpha,\beta)^T}=\{(f,g)^T\in\mathbf{H}\mid g(1)=0\}, \tag{29}$$

then on $S_{(\alpha,\beta)^T}$, the system $(21)-(24)$ becomes

$$\alpha_t(x,t)=-\varepsilon_1(x)\alpha_x(x,t) \tag{30}$$

$$\beta_t(x,t)=\varepsilon_2(x)\beta_x(x,t) \tag{31}$$

$$\alpha(0,t)=q\beta(0,t) \tag{32}$$

$$\beta(1,t)=0, \tag{33}$$

and we can obtain that

$$\dot{E}(t)\le -aE(t), \tag{34}$$

where

$$a=\frac{1}{2}\min_{x\in[0,1]}\{\varepsilon_1(x),\varepsilon_2(x)\}>0. \tag{35}$$

The following lemma can then be proved.

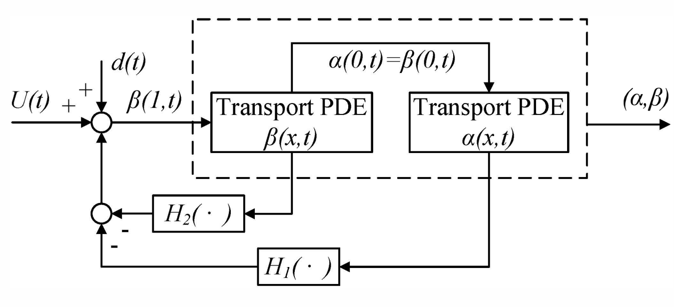


Fig. 2. Block diagram of systems $(21)-(24)$ (H_1 and H_2 are operators, of which the meaning should be clear from (24).)

Lemma 1: For any initial data $(\alpha(\cdot,0),\beta(\cdot,0))^T \in S_{(\alpha,\beta)^T}$, there exists a unique (mild) solution to $(30)-(33)$ such that

$$(\alpha(\cdot,t),\beta(\cdot,t))^T \in C([0,\infty);\mathbf{H}). \tag{36}$$

Moreover, the system $(21)-(24)$ is exponentially stable in $S_{(\alpha,\beta)^T}$:

$$\|(\alpha(\cdot,t),\beta(\cdot,t))^T\|_{\mathbf{H}} \leqslant e^{-a/2t}\|(\alpha(\cdot,0),\beta(\cdot,0))^T)\|_{\mathbf{H}}. \tag{37}$$

Transforming $S_{(\alpha,\beta)^T}$ through $(5)-(6)$, that is,

$$\begin{aligned} S_{(u,v)^T}(t) =& v(1,t) - \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi \\ &- \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi, \end{aligned} \tag{38}$$

we get the sliding surface for the system $(1)-(4)$ as

$$\begin{aligned} S_{(u,v)^T} =& \Big\{(f,g)^T \in \mathbf{H} \mid g(1) - \int_0^1 K^{vu}(1,\xi)f(\xi)d\xi \\ &- \int_0^1 K^{vv}(1,\xi)g(\xi)d\xi = 0\Big\}, \end{aligned} \tag{39}$$

on which the original system $(1)-(4)$ becomes

$$u_t(x,t) = -\varepsilon_1(x)u_x(x,t) + c_1(x)v(x,t) \tag{40}$$

$$v_t(x,t) = \varepsilon_2(x)v_x(x,t) + c_2(x)u(x,t) \tag{41}$$

$$u(0,t) = qv(0,t) \tag{42}$$

$$v(1,t) = \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi, \tag{43}$$

which is also exponentially stable by (37) and the equivalence between $(5)-(6)$ and $(13)-(14)$. Thus, the following theorem can be presented.

Theorem 1: For any initial data $(u(\cdot,0),v(\cdot,0))^T \in S_{(u,v)^T}$, there exists a unique (mild) solution to $(40)-(43)$ such that

$$(u(\cdot,t),v(\cdot,t))^T \in C([0,\infty);\mathbf{H}). \tag{44}$$

Moreover, the system $(1)-(4)$ is exponentially stable in $S_{(u,v)^T}$, i.e., there exists $M_S > 0$ such that

$$\|(u(\cdot,t),v(\cdot,t))^T\|_{\mathbf{H}} \leqslant M_S e^{-a/2t}\|(u(\cdot,0),v(\cdot,0))^T)\|_{\mathbf{H}}. \tag{45}$$

B. Reaching condition

Differentiating (38) with respect to t, we get

$$\begin{aligned} \dot{S}_{(u,v)^T}(t) =& \dot{U}(t) + \dot{d}(t) - \int_0^1 K^{vu}(1,\xi)u_t(\xi,t)d\xi \\ &- \int_0^1 K^{vv}(1,\xi)v_t(\xi,t)d\xi. \end{aligned} \tag{46}$$

If choosing a sliding mode controller such that

$$\begin{aligned} \dot{U}(t) =& \int_0^1 K^{vu}(1,\xi)u_t(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v_t(\xi,t)d\xi \\ &- K\frac{S_{(u,v)^T}(t)}{|S_{(u,v)^T}(t)|} \quad \text{for} \quad S_{(u,v)^T}(t) \neq 0, \end{aligned} \tag{47}$$

where $K > M$, then

$$\dot{S}_{(u,v)^T}(t) = \dot{d}(t) - K\frac{S_{(u,v)^T}(t)}{|S_{(u,v)^T}(t)|} \quad \text{for} \quad S_{(u,v)^T}(t) \neq 0. \tag{48}$$

Thus the following holds:

$$\begin{aligned} \frac{d}{dt}|S_{(u,v)^T}(t)|^2 =& 2Re\left(\overline{S_{(u,v)^T}(t)}\dot{S}_{(u,v)^T}(t)\right) \\ =& 2Re\left(\overline{S_{(u,v)^T}(t)}\dot{d}(t)\right) - 2K|S_{(u,v)^T}(t)| \\ \le& -2(K-M)|S_{(u,v)^T}(t)|, \end{aligned} \tag{49}$$

which is the finite time "reaching condition" for the system $(21)-(24)$. Existence of $\dot{S}$ is to be proved rigorously in Section IV.

C. Sliding mode controller

Choose the sliding mode boundary controller as

$$\begin{aligned} U(t) =& \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi \\ &- K\int_0^t \frac{S_{(u,v)^T}(\tau)}{|S_{(u,v)^T}(\tau)|}d\tau \quad \text{for} \quad S_{(u,v)^T}(t) \neq 0, \end{aligned} \tag{50}$$

then the resulting closed-loop system is

$$u_t(x,t) = -\varepsilon_1(x)u_x(x,t) + c_1(x)v(x,t) \tag{51}$$

$$v_t(x,t) = \varepsilon_2(x)v_x(x,t) + c_2(x)u(x,t) \tag{52}$$

$$u(0,t) = qv(0,t) \tag{53}$$

$$\begin{aligned} v(1,t) =& \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi \\ &- K\int_0^t \frac{S_{(u,v)^T}(\tau)}{|S_{(u,v)^T}(\tau)|}d\tau + d(t) \quad \text{for} \quad S_{(u,v)^T}(t) \neq 0. \end{aligned} \tag{54}$$

From transformation $(13)-(14)$, the corresponding controller for the system $(21)-(24)$ is

$$\begin{aligned} U(t) =& \int_0^1 \alpha(\xi,t)\left(K^{vu}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\alpha}(\eta,\xi)d\eta \right. \\ &\left. + \int_\xi^1 K^{vv}(1,\eta)L^{\beta\alpha}(\eta,\xi)d\eta\right)d\xi \\ &+ \int_0^1 \beta(\xi,t)\left(K^{vv}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\beta}(\eta,\xi)d\eta \right. \\ &\left. + \int_\xi^1 K^{vv}(1,\eta)L^{\beta\beta}(\eta,\xi)d\eta\right)d\xi. \\ &- K\int_0^t \frac{\beta(1,\tau)}{|\beta(1,\tau)|}d\tau \quad \text{for} \quad S_{(\alpha,\beta)^T}(t) \neq 0, \end{aligned} \tag{55}$$

and thus the corresponding closed-loop control $(\alpha,\beta)^T$-systems are

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \tag{56}$$

$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \tag{57}$$

$$\alpha(0,t) = q\beta(0,t) \tag{58}$$

$$\beta(1,t) = d(t) - K\int_0^t \frac{\beta(1,\tau)}{|\beta(1,\tau)|}d\tau \triangleq \tilde{d}(t) \quad \text{for} \quad S_{(\alpha,\beta)^T}(t) \neq 0. \tag{59}$$

IV. Solution of Closed-Loop Systems

Define an operator $\mathscr{A}: D(\mathscr{A})(\subset \mathbf{H}) \to \mathbf{H}$ as follows:

$$\mathscr{A}(f,g)^T = \left(-\varepsilon_1(x)f', \ \ \varepsilon_2(x)g'\right)^T, \ \forall (f,g)^T \in D(\mathscr{A}), \tag{60}$$

$$D(\mathscr{A}) = \{(f,g)^T \in \left(H^1(0,1)\right)^2 \ | \ f(0)=qg(0), g(1)=0\}, \tag{61}$$

of which the adjoint operator is

$$\begin{aligned}&\mathscr{A}^*(\phi,\psi)^T\\ &= \left(\varepsilon_1(x)\left(\phi' + \frac{\phi}{x-2}\right), -\varepsilon_2(x)\left(\psi' + \frac{\psi}{x+1}\right)\right)^T,\\ &\quad \forall (\phi,\psi)^T \in D(\mathscr{A}^*),\end{aligned} \tag{62}$$

$$D(\mathscr{A}^*) = \{(\phi,\psi)^T \in \left(H^1(0,1)\right)^2 \ | \ \phi(0) = q\psi(0), \phi(1)=0\}. \tag{63}$$

Lemma 2: $\mathscr{A}^{-1}$ exists and is compact on $\mathbf{H}$. Moreover, $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative in $\mathbf{H}$, and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathbf{H}$.

Proof: For any given $(f,g)^T \in \mathbf{H}$, solve

$$\mathscr{A}(f_1,g_1)^T = \left(-\varepsilon_1(x)f_1', \ \ \varepsilon_2(x)g_1'\right)^T = (f,g)^T, \tag{64}$$

$$f_1(0) = qg_1(0), g_1(1) = 0 \tag{65}$$

to get

$$f_1(x) = q\int_1^0 \frac{g(\xi)}{\varepsilon_2(\xi)}d\xi - \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)}d\xi, \tag{66}$$

$$g_1(x) = \int_1^x \frac{g(\xi)}{\varepsilon_2(\xi)}d\xi, \tag{67}$$

which is the unique solution $(f_1,g_1)^T \in D(\mathscr{A})$. Hence, $\mathscr{A}^{-1}$ exists and is compact on $\mathbf{H}$ by the Sobolev embedding theorem.

Let $(f,g)^T \in D(\mathscr{A})$ and $(\phi,\psi)^T \in D(\mathscr{A}^*)$, then

$$\begin{aligned}&Re < \mathscr{A}(f,g)^T, \ \ (f,g)^T >_{\mathbf{H}}\\ &= -\frac{1}{2}|f(1)|^2 - \frac{1}{2}\int_0^1 (|f(x)|^2 + 2q^2|g(x)|^2)dx \le 0,\end{aligned} \tag{68}$$

$$\begin{aligned}&Re < (\phi,\psi)^T, \ \ \mathscr{A}^*(\phi,\psi)^T >_{\mathbf{H}}\\ &= -2q^2|\psi(1)|^2 - \frac{1}{2}\int_0^1 (|\phi(x)|^2 + 2q^2|\psi(x)|^2)dx \le 0.\end{aligned} \tag{69}$$

Hence $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative in $\mathbf{H}$, and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathbf{H}$ by the Lumer-Philips theorem.

■

For any $(\phi,\psi)^T \in D(\mathscr{A}^*)$, we can get from (56) – (59) that

$$\begin{aligned}&\frac{d}{dt}\left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_{\mathbf{H}} = \left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \mathscr{A}^*\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_{\mathbf{H}}\\ &+ \left\langle \begin{pmatrix}0\\ 4q^2\delta(x-1)\end{pmatrix}\tilde{d}(t), \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_{D(\mathscr{A}^*)'\times D(\mathscr{A}^*)},\end{aligned} \tag{70}$$

then the system (56) – (59) can be written as follows:

$$\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mathscr{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \mathscr{B}\tilde{d}(t), \tag{71}$$

$$\mathscr{B} = \begin{pmatrix}0\\ 4q^2\delta(x-1)\end{pmatrix}, \tag{72}$$

where $\delta(\cdot)$ denotes the Dirac distribution.

Lemma 3: $\mathscr{B}$ is admissible for the C_0-semigroup $e^{\mathscr{A}t}$ generated by $\mathscr{A}$.

Proof: Consider the observation problem of dual system of (71) – (72):

$$\frac{d}{dt}\begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix} = \mathscr{A}^*\begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix} \tag{73}$$

$$y^* = \mathscr{B}^*\begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix}. \tag{74}$$

(Part One) Differentiate

$$\begin{aligned}&E_{(\alpha^*,\beta^*)^T}(t)\\ &= \frac{1}{2}\int_0^1 \left(\frac{2-x}{\varepsilon_1(x)}|\alpha^*(x,t)|^2 + \frac{2q^2(1+x)}{\varepsilon_2(x)}|\beta^*(x,t)|^2\right)dx,\end{aligned} \tag{75}$$

with respect to t along the solution of (73) – (74), then we can get

$$\begin{aligned}\dot{E}_{(\alpha^*,\beta^*)^T}(t) = &-2q^2|\beta^*(1,t)|^2\\ &- \frac{1}{2}\int_0^1 \left(|\alpha^*(x,t)|^2 + 2q^2|\beta^*(x,t)|^2\right)dx\\ &\le 0,\end{aligned} \tag{76}$$

and hence

$$E_{(\alpha^*,\beta^*)^T}(T) \le E_{(\alpha^*,\beta^*)^T}(0), \quad \forall T > 0. \tag{77}$$

By some lengthy calculation, we can get that for any $T > 0$,

$$\begin{aligned}&\int_0^T |y^*(t)|^2 dt = 16q^4\int_0^T |\beta^*(1,t)|^2 dt\\ &\le \frac{16q^2}{\varepsilon_2(1)}\max_{x\in[0,1]}\{\varepsilon_2(x)\}\\ &\times\left(T\max_{x\in[0,1]}\{|\frac{1-x}{x+1}\varepsilon_2(x) + x\varepsilon_2'(x)|\} + \frac{1}{2}\right)E_{(\alpha^*,\beta^*)^T}(0).\end{aligned} \tag{78}$$

(Part Two) A direct computation shows

$$\mathscr{A}^{*-1}(\phi,\psi)^T = (\phi_1,\psi_1)^T \tag{79}$$

$$\phi_1(x) = \frac{1}{2-x}\int_x^1 \frac{\phi(\eta)}{\varepsilon_1(\eta)}(\eta-2)d\eta \tag{80}$$

$$\begin{aligned}\psi_1(x) = &\frac{1}{2q(x+1)}\int_0^1 \frac{\phi(\eta)}{\varepsilon_1(\eta)}(\eta-2)d\eta\\ &- \frac{1}{1+x}\int_0^x \frac{\psi(\eta)}{\varepsilon_2(\eta)}(1+\eta)d\eta,\end{aligned} \tag{81}$$

and

$$\begin{aligned}\mathscr{B}^*\mathscr{A}^{*-1}(\phi,\psi)^T = &q\int_0^1 \frac{\phi(\eta)}{\varepsilon_1(\eta)}(\eta-2)d\eta\\ &-2q^2\int_0^1 \frac{\psi(\eta)}{\varepsilon_2(\eta)}(1+\eta)d\eta.\end{aligned} \tag{82}$$

Hence, $\mathscr{B}^*\mathscr{A}^{*-1}$ is bounded on $\mathbf{H}$.

Results from Part One and Part Two show that $\mathscr{B}^*$ is admissible for the C_0-semigroup $e^{\mathscr{A}^* t}$ generated by $\mathscr{A}^*$, and so is $\mathscr{B}$ for $e^{\mathscr{A} t}$. ■

Therefore, if for some $T > 0$, $S_{(\alpha,\beta)^T} \in C[0,T]$ and $S_{(\alpha,\beta)^T}(t) \neq 0$, $\forall\ t \in [0,T]$, then for any initial data $(\alpha(\cdot,0),\beta(\cdot,0))^T \in \mathbf{H}$, systems $(71)-(72)$ admit a unique solution $(\alpha(\cdot,t),\beta(\cdot,t))^T \in C([0,T];\mathbf{H})$.

Suppose that for some $T > 0$, $S_{(\alpha,\beta)^T}(t) \neq 0$ for all $t \in [0,T]$. Take the inner product with $(\phi,\psi)^T = (0,x)^T \in D(\mathscr{A}^*)$ on both sides of (71) to get the left hand side as

$$\begin{aligned}\frac{d}{dt}\int_0^1 \frac{2q^2(1+x)}{\varepsilon_2(x)}\beta(x,t)\bar{x}dx &= \int_0^1 2q^2(1+x)\beta_x(x,t)xdx \\ &= 4q^2\beta(1,t) - 2q^2\int_0^1 \beta(x,t)(1+2x)dx, \forall t \in [0,T]\ a.e.,\end{aligned} \tag{83}$$

and the right hand side as

$$\begin{aligned}&< (\alpha,\beta)^T, \mathscr{A}^*(0,x)^T > + < (0, 4q^2\delta(x-1)\tilde{d}(t))^T, (0,x)^T > \\ &= -\int_0^1 2q^2\beta(x,t)(1+2x)dx + 4q^2\tilde{d}(t), \forall t \in [0,T]\ a.e..\end{aligned} \tag{84}$$

Since $q \neq 0$, we have

$$\beta(1,t) = \tilde{d}(t), \forall t \in [0,T]\ a.e., \tag{85}$$

that is,

$$\dot{S}_{(\alpha,\beta)^T}(t) = \dot{\tilde{d}}(t) = \dot{d}(t) - K\frac{S_{(\alpha,\beta)^T}(t)}{|S_{(\alpha,\beta)^T}(t)|}, \forall t \in [0,T]\ a.e., \tag{86}$$

which is equivalent to (48).

Lemma 4: There exists a unique, continuous, nonzero solution to (86) on some interval $[0, T_{\max}]$.

Proof: Suppose that for some $T_0 \geq 0$, $S_{(\alpha,\beta)^T}(T_0) = S_0 \neq 0$, then (86) is equivalent to:

$$S_{(\alpha,\beta)^T}(t) = S_0 + \int_{T_0}^t \dot{d}(\tau)d\tau - K\int_{T_0}^t \frac{S_{(\alpha,\beta)^T}(\tau)}{|S_{(\alpha,\beta)^T}(\tau)|}d\tau,\ \forall t \geq T_0. \tag{87}$$

Define a closed subspace of $C\left[T_0, T_0 + \frac{|S_0|}{3(M+K)}\right]$ by

$$\begin{aligned}\Omega = \Big\{ S \in C\left[T_0, T_0 + \frac{|S_0|}{3(M+K)}\right] \ \Big|\ & S(T_0) = S_0, \\ |S(t)| \geq \frac{2}{3}|S_0|,\ \forall\ t \in & \left[T_0, T_0 + \frac{|S_0|}{3(M+K)}\right] \Big\},\end{aligned} \tag{88}$$

and define a mapping F on Ω by

$$(FS)(t) = S_0 + \int_{T_0}^t \dot{d}(\tau)d\tau - K\int_{T_0}^t \frac{S(\tau)}{|S(\tau)|}d\tau, \tag{89}$$

then $\forall S \in \Omega$, $\forall\ t \in \left[T_0, T_0 + \frac{|S_0|}{3(M+K)}\right]$, we have

$$|(FS)(t)| \geq |S_0| - (M+K)(t-T_0) \geq \frac{2}{3}|S_0|, \tag{90}$$

that is, $F\Omega \subset \Omega$. Since

$$\begin{aligned}|(FS_1)(t) - (FS_2)(t)| \leq & 2K\int_{T_0}^t \frac{|S_1(\tau) - S_2(\tau)|}{|S_1(\tau)|}d\tau \\ \leq & \frac{K}{M+K}\|S_1 - S_2\|_\Omega,\end{aligned} \tag{91}$$

where $\|S\|_\Omega = \|S\|_{C\left[T_0, T_0 + \frac{|S_0|}{3(M+K)}\right]}$, then the mapping F is a contraction mapping on Ω. By the Banach fixed point theorem, the proof can be completed. ■

Thus, the following lemma is obtained.

Lemma 5: Suppose that d and $\dot{d}$ are bounded measurable in time, then for any initial data $(\alpha(\cdot,0),\beta(\cdot,0))^T \in \mathbf{H}$, there exists $T_{\max} \geq 0$, depending on initial data, such that the system $(56)-(59)$ admits a unique solution

$$(\alpha(\cdot,t),\beta(\cdot,t))^T \in C([0,T_{\max}];\mathbf{H}), \tag{92}$$

and $\beta(1,t) = 0$ for all $t \geq T_{\max}$. Moreover, $S_{(\alpha,\beta)^T}(t) = \beta(1,t)$ is continuous and monotone in $[0, T_{\max}]$.

By the equivalence between $(5)-(6)$ and $(13)-(14)$, the following main theorem can be proved.

Theorem 2: Suppose that d and $\dot{d}$ is bounded measurable in time, then for any initial data $(u(\cdot,0),v(\cdot,0))^T \in \mathbf{H}$, there exists $T_{\max} \geq 0$, depending on initial data, such that the system $(51)-(54)$ admits a unique solution

$$(u(\cdot,t),v(\cdot,t))^T \in C([0,T_{\max}];\mathbf{H}) \tag{93}$$

and

$$\begin{aligned}S_{(u,v)^T}(t) = & v(1,t) - \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi \\ & - \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi = 0\end{aligned} \tag{94}$$

for all $t \geq T_{\max}$. Moreover, $S_{(u,v)^T}(t)$ is continuous and monotone in $[0, T_{\max}]$. On the sliding mode surface $S_{(u,v)^T}(t) = 0$, the system $(1)-(4)$ becomes $(40)-(43)$, which is exponentially stable.

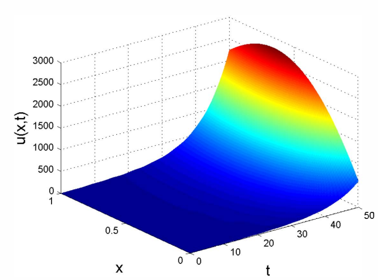
V. Numerical simulation

Consider the system $(1)-(4)$ with $\varepsilon_1(x) = 0.1$, $\varepsilon_2(x) = 0.2$, $c_1(x) = 0.3$, $c_2(x) = 0.4$, $q = 1/4$ and $d(t) = 10\sin t$. Set the initial data as $u(x,0) = \frac{5}{2}(1-x)$, $v(x,0) = 10(1-x)$.

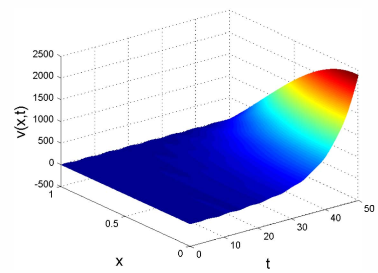
Take the time length, steps of time and space as 50, 0.01 and 0.01, then open-loop system response and and closed-loop system response with sliding mode controller (choosing $\varepsilon = 0.001$) are shown in Fig. 3 and Fig. 4, respectively. As can be seen from these figures, although the open-loop system blows up, the designed controller can stabilize it. It's also worth noting that chattering is avoided in the closed-loop control system.

VI. Conclusions and future work

In this paper, a 2×2 system of first-order linear hyperbolic PDEs subject to boundary input disturbance is stabilized by sliding mode control approach. Disturbance rejection and finite time stability is achieved for the resulting closed-loop control system.

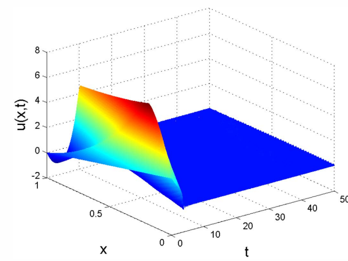


(a)

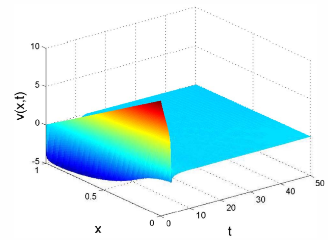


(b)

Fig. 3. Simulation results for the open-loop system



(a)



(b)

Fig. 4. Simulation results for the closed-loop control system

Firstly, the employed first-order (and thus non-chattering) sliding mode controller is effective only when the disturbance has bounded derivative. For the case of unbounded disturbance derivatives, some other methods and techniques might be needed. Secondly, sliding mode control has the restriction that it does not work for non-matched disturbance, and one possible future work would be stabilization of linear 2×2 hyperbolic systems with non-matched boundary output disturbances. Thirdly, only the case of $q \neq 0$ has been considered in this paper. However, for the case of $q = 0$, we may expect that similar results (see, [7]) could be obtained.

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