Systems & Control Letters 60 (2011) 540–545

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# State and output feedback boundary control for a coupled PDE–ODE system☆

Shuxia Tang, Chengkang Xie *

*School of Mathematics and System Science, Southwest University, Chongqing 400715, China*

ARTICLE INFO

*Article history:*
Received 2 June 2010
Received in revised form
24 December 2010
Accepted 20 April 2011
Available online 24 May 2011

*Keywords:*
Coupled system
PDEs
Boundary control
Output feedback

ABSTRACT

This note is devoted to stabilizing a coupled PDE–ODE system with interaction at the interface. First, a state feedback boundary controller is designed, and the system is transformed into an exponentially stable PDE–ODE cascade with an invertible integral transformation, where PDE backstepping is employed. Moreover, the solution to the resulting closed-loop system is derived explicitly. Second, an observer is proposed, which is proved to exhibit good performance in estimating the original coupled system, and then an output feedback boundary controller is obtained. For both the state and output feedback boundary controllers, exponential stability analyses in the sense of the corresponding norms for the resulting closed-loop systems are provided. The boundary controller and observer for a scalar coupled PDE–ODE system as well as the solutions to the closed-loop systems are given explicitly.



## 1. Introduction

In control engineering, topics concerning coupled systems are popular, which have rich physical backgrounds such as coupled electromagnetic, coupled mechanical, and coupled chemical reactions. Many results on controllability of the coupled PDE–PDE systems have been achieved such as in [1–3]. Applicable controllers of state and output feedback for coupled PDE–PDE systems as well as coupled PDE–ODE systems, however, are original areas. As a beginning, control design of cascaded PDE–ODE systems were considered in [4–9], where through decoupling and PDE backstepping, boundary controllers were successfully established.

The system considered in this note is a coupled PDE–ODE system with interaction between the ODE and the PDE. At the interface, the ODE acts back on the PDE at the same time as the PDE acts on the ODE. It models the solid–gas interaction of heat diffusion and chemical reaction, where the interaction occurs at the interface; see Fig. 1. This system is certainly more complex than just a single ODE or a single PDE, and even more complex than a cascade of PDE and ODE, in which only the PDE acts on the ODE, or only the ODE acts on the PDE. Thus, it is needed to overcome some difficulties in control design. Some special techniques and PDE backstepping are used to develop controllers.

This note is organized as follows. In Section 2, the problem is formulated. In Section 3, a state feedback boundary controller is designed to stabilize the coupled PDE–ODE system. In Section 4, an observer, as well as an output feedback boundary controller, is designed. And a scalar coupled PDE–ODE system is given in Section 5 as an example, where the controller, the observer and solutions to the closed-loop systems are obtained explicitly. In Section 6, some comments are made on the coupled PDE–ODE systems.

## 2. Problem formulation

Consider the following coupled PDE–ODE system

$$\dot{X}(t) = AX(t) + Bu_x(0,t) \tag{1}$$

$$u_t(x,t) = u_{xx}(x,t), \quad x \in (0,l) \tag{2}$$

$$u(0,t) = CX(t) \tag{3}$$

$$u(l,t) = U(t) \tag{4}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, the pair $(A, B)$ is assumed to be stabilizable, $u(x,t) \in \mathbb{R}$ is the PDE state, $C^T$ is a constant vector, and $U(t)$ is the scalar input to the entire system. The coupled system is depicted in Fig. 2. The control objective is to exponentially stabilize the system signal $(X(t), u(x,t))$.

## 3. State feedback controller

Admittedly, if an invertible transformation $(X,u) \mapsto (X,w)$ can be sought to transform the system (1)–(4) into an exponentially stable target system companied with a controller, e.g., the following system

$$\dot{X}(t) = (A+BK)X(t) + Bw_x(0,t) \tag{5}$$

$$w_t(x,t) = w_{xx}(x,t), \quad x \in (0,l) \tag{6}$$

☆ This work is supported by the Fundamental Research Funds for the Central Universities under contract XDJK2009C099 and National Nature Science Foundation under Grant 11026111.

* Corresponding author. Tel.: +86 13637769120.
*E-mail addresses:* tkongzhi@swu.edu.cn (S. Tang), cxie@swu.edu.cn (C. Xie).

0167-6911/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2011.04.011

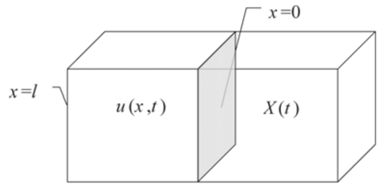


**Fig. 1.** A physical background of the coupled system.

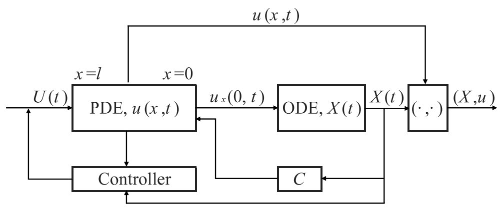


**Fig. 2.** Control configuration of the coupled system.

$$w(0,t)=0 \tag{7}$$

$$w(l,t)=0 \tag{8}$$

where $K$ is chosen such that $A+BK$ is Hurwitz, then, exponential stabilization of the original closed-loop system can be achieved. Here the transformation $(X,u) \mapsto (X,w)$ is postulated in the following form

$$X(t)=X(t) \tag{9}$$

$$w(x,t)=u(x,t)-\int_0^x \kappa(x,y)u(y,t)\mathrm{d}y-\Phi(x)X(t) \tag{10}$$

where the gain functions $\kappa(x,y)\in\mathbb{R}$ and $\Phi(x)^T\in\mathbb{R}^n$ are to be determined.

The partial derivatives of $w(x,t)$ with respect to $x$ are given by

$$\begin{aligned} w_x(x,t) &= u_x(x,t)-\kappa(x,x)u(x,t) \\ &\quad -\int_0^x \kappa_x(x,y)u(y,t)\mathrm{d}y-\Phi'(x)X(t) \end{aligned} \tag{11}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t)-\kappa(x,x)u_x(x,t) \\ &\quad -\left(\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)+\kappa_x(x,x)\right)u(x,t) \\ &\quad -\int_0^x \kappa_{xx}(x,y)u(y,t)\mathrm{d}y-\Phi''(x)X(t) \end{aligned} \tag{12}$$

where the notation $\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)=\kappa_x(x,x)+\kappa_y(x,x)$ is used. The derivative of $w(x,t)$ with respect to $t$ is

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t)-\kappa(x,x)u_x(x,t)+\kappa_y(x,x)u(x,t) \\ &\quad -\int_0^x \kappa_{yy}(x,y)u(y,t)\mathrm{d}y \\ &\quad +(\kappa(x,0)-\Phi(x)B)u_x(0,t)-\kappa_y(x,0)u(0,t) \\ &\quad -\Phi(x)AX(t). \end{aligned} \tag{13}$$

Setting $x=0$ in the Eqs. (10) and (11), and by (12) and (13), the following equations hold:

$$w(0,t)=(C-\Phi(0))X(t)$$

$$w_x(0,t)=u_x(0,t)-(\Phi'(0)+\kappa(0,0)C)X(t)$$

$$\begin{aligned} w_t(x,t)-w_{xx}(x,t) &= 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)\right)u(x,t) \\ &\quad +\int_0^x(\kappa_{xx}(x,y)-\kappa_{yy}(x,y))u(y,t)\mathrm{d}y \\ &\quad +(\kappa(x,0)-\Phi(x)B)\,u_x(0,t) \\ &\quad +(\Phi''(x)-\Phi(x)A-\kappa_y(x,0)C)X(t) \end{aligned}$$

where the fact that $u(0,t)=CX(t)$ is used. To satisfy (5)–(7), it is sufficient that $\kappa(x,y)$ and $\Phi(x)$ satisfy

$$\kappa_{xx}(x,y)=\kappa_{yy}(x,y) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)=0, \qquad \kappa(x,0)=\Phi(x)B \tag{15}$$

and

$$\Phi''(x)-\Phi(x)A-\kappa_y(x,0)C=0 \tag{16}$$

$$\Phi(0)=C, \qquad \Phi'(0)=K-\kappa(0,0)C. \tag{17}$$

Although the PDE (14)–(15) and the ODE (16)–(17) are still coupled, they can be decoupled and solved explicitly through some techniques of algebra and analytical mathematics.

First, the solution to the PDE (14)–(15) is

$$\kappa(x,y)=\Phi(x-y)B. \tag{18}$$

Second, substituting (18) into (16) and (17) respectively, it is obtained that

$$\Phi''(x)+\Phi'(x)BC-\Phi(x)A=0 \tag{19}$$

and

$$\Phi'(0)=K-\Phi(0)BC=K-CBC.$$

Let $I$ be an identity matrix, then the explicit solution to the ODE (16)–(17) is obtained:

$$\Phi(x)=\begin{pmatrix}C & K-CBC\end{pmatrix}e^{Dx}\begin{pmatrix}I\\0\end{pmatrix}$$

where

$$D=\begin{pmatrix}0 & A\\ I & -BC\end{pmatrix}$$

and the explicit solution to the PDE (14)–(15) is

$$\kappa(x,y)=\begin{pmatrix}C & K-CBC\end{pmatrix}e^{D(x-y)}\begin{pmatrix}I\\0\end{pmatrix}B.$$

The integral transformation $(X,u)\mapsto(X,w)$ defined by (9)–(10) is invertible. Suppose the inverse transformation $(X,w)\mapsto(X,u)$ as the following form

$$X(t)=X(t) \tag{20}$$

$$u(x,t)=w(x,t)+\int_0^x \iota(x,y)w(y,t)\mathrm{d}y+\Psi(x)X(t) \tag{21}$$

where the kernel functions $\iota(x,y)\in\mathbb{R}$ and $\Psi(x)^T\in\mathbb{R}^n$ are to be determined.

Following the same procedure of determination of the kernels $\kappa(x,y)$ and $\Phi(x)$, compute the derivatives $u_x$, $u_{xx}$ and $u_t$, and a sufficient condition for $\iota(x,y)$ and $\Psi(x)$ to satisfy (1)–(3) is obtained as

$$\iota_{xx}(x,y)=\iota_{yy}(x,y) \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\iota(x,x)=0, \qquad \iota(x,0)=\Psi(x)B \tag{23}$$

and

$$\Psi''(x)-\Psi(x)(A+BK)=0 \tag{24}$$

$$\Psi(0)=C, \qquad \Psi'(0)=K. \tag{25}$$

This cascade system can also be solved explicitly. First, the explicit solution to the ODE (24)–(25) is

$$\Psi(x)=\begin{pmatrix}C & K\end{pmatrix}e^{Ex}\begin{pmatrix}I\\0\end{pmatrix}$$

where

$$E=\begin{pmatrix}0 & A+BK\\ I & 0\end{pmatrix}.$$

Second, the explicit solution to the PDE (22)–(23) is

$$\iota(x,y)=\Psi(x-y)B=\begin{pmatrix}C & K\end{pmatrix}e^{E(x-y)}\begin{pmatrix}I\\0\end{pmatrix}B.$$

Thus, the direct and inverse transformations are written as

$$w(x,t) = u(x,t) - \int_0^x \Phi(x-y)u(y,t)\mathrm{d}yB - \Phi(x)X(t) \quad (26)$$

$$u(x,t) = w(x,t) + \int_0^x \Psi(x-y)w(y,t)\mathrm{d}yB + \Psi(x)X(t). \quad (27)$$

Evaluating (26) at $x = l$, and by the boundary condition (4) and (8), a controller is obtained as follows:

$$U(t) = \int_0^l \Phi(l-y)u(y,t)\mathrm{d}yB + \Phi(l)X(t). \quad (28)$$

Furthermore, the explicit solution to the closed-loop system (1)–(4) under the controller (28) can also be obtained if the initial state $(X(0), u(x,0))$ is known. First, the solution to the heat equation (6)–(8) is

$$w(x,t) = \frac{2}{l}\sum_{m=1}^{\infty} e^{-\frac{m^2\pi^2}{l^2}t}\sin\left(\frac{m\pi}{l}x\right)\mu_m \quad (29)$$

where

$$\mu_m = \int_0^l w_0(\xi)\sin\left(\frac{m\pi}{l}\xi\right)\mathrm{d}\xi \quad (30)$$

and the initial condition $w_0(x)$ is calculated by the initial state $u(x,0)$ through (26). Second, signal $X(t)$ is obtained by

$$X(t) = X(0)e^{(A+BK)t} + \int_0^t e^{(A+BK)(t-\tau)}Bw_x(0,\tau)\mathrm{d}\tau \quad (31)$$

and signal $u(x,t)$ is obtained from (21).

**Theorem 1.** *For any initial data $X(0) \in \mathbb{R}$ and $u(\cdot,0) \in H^1(0,l)$, the closed-loop system consisting of the plant (1)–(4) and the control law (28) has a classical solution, which is exponentially stabilized in the sense of the norm*

$$\|(X(t), u(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)}$$

*where $|\cdot|$ denotes the Euclidean norm.*

**Proof.** Consider the following Lyapunov function

$$V(t) = X^TPX + \frac{a}{2}\|w(\cdot,t)\|^2_{L^2(0,l)} + \frac{1}{2}\|w_x(\cdot,t)\|^2_{L^2(0,l)}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^TP = -Q$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later.

For simplicity, in the following proof, the symbol $\|\cdot\|$ denotes the $L^2(0,l)$ norm. From the backstepping transformations (26) and (27), it can be obtained that

$$\|w\|^2 \le \alpha_1\|u\|^2 + \alpha_2|X|^2 \quad (32)$$

$$\|u\|^2 \le \beta_1\|w\|^2 + \beta_2|X|^2 \quad (33)$$

$$\|w_x\|^2 \le \alpha_3\|u_x\|^2 + \alpha_4\|u\|^2 + \alpha_5|X|^2 \quad (34)$$

$$\|u_x\|^2 \le \beta_3\|w_x\|^2 + \beta_4\|w\|^2 + \beta_5|X|^2 \quad (35)$$

where

$$\alpha_1 = 3(1 + l\|\Phi B\|^2), \qquad \alpha_2 = 3\|\Phi\|^2$$
$$\beta_1 = 3(1 + l\|\Psi B\|^2), \qquad \beta_2 = 3\|\Psi\|^2$$
$$\alpha_3 = 4, \qquad \alpha_4 = 4((CB)^2 + l\|\Phi_x B\|^2),$$
$$\alpha_5 = 4\|\Phi'\|^2$$
$$\beta_3 = 4, \qquad \beta_4 = 4((CB)^2 + l\|\Psi_x B\|^2), \qquad \beta_5 = 4\|\Psi'\|^2.$$

Then, it can be obtained that

$$\underline{\delta}(|X|^2 + \|u\|^2_{H^1(0,l)}) \le V \le \overline{\delta}(|X|^2 + \|u\|^2_{H^1(0,l)})$$

where

$$\overline{\delta} = \max\left\{\lambda_{\max}(P) + \frac{a\alpha_2}{2} + \frac{\alpha_5}{2}, \frac{a\alpha_1}{2} + \frac{\alpha_4}{2}, \frac{\alpha_3}{2}\right\}$$

$$\underline{\delta} = \frac{\min\left\{\frac{a}{2}, \frac{1}{2}, \lambda_{\min}(P)\right\}}{\max\{\beta_3, \beta_1+\beta_4, \beta_2+\beta_5+1\}}.$$

Calculate the derivative of the Lyapunov function along the solutions to the system (5)–(8), then

$$\dot{V} \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 + 2\frac{|PB|^2}{\lambda_{\min}(Q)}w_x(0,t)^2 - a\|w_x\|^2 - \|w_{xx}\|^2.$$

By Agmon's inequality, it can be proved that the following inequality holds:

$$-\|w_{xx}\|^2 \le \frac{1+l}{l}\|w_x\|^2 - w_x(0,t)^2.$$

Thus

$$\dot{V} \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - 2\frac{|PB|^2}{\lambda_{\min}(Q)} - \frac{1+l}{l}\right)\|w_x\|^2 - w_x(0,t)^2.$$

Now take

$$a > 2\frac{|PB|^2}{\lambda_{\min}(Q)} + \frac{1+l}{l}$$

then by Poincaré inequality, it can be shown that

$$\dot{V} \le -bV$$

where

$$b = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{2}{1+4l^2}\left(1 - 2\frac{|PB|^2}{a\lambda_{\min}(Q)} - \frac{1+l}{al}\right)\right\}.$$

Therefore,

$$V(t) \le V(0)e^{-bt}.$$

Let $\delta = \overline{\delta}/\underline{\delta}$, then

$$|X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)} \le \delta(|X(0)|^2 + \|u(\cdot,0)\|^2_{H^1(0,l)})e^{-bt}$$

holds for all $t \ge 0$, which completes the proof. □

## 4. Observer and output feedback controller

Assume that only $u_x(0,t)$ is available for measurement, or for economic consideration, only $u_x(0,t)$ is measured. To manipulate a control for the system (1)–(4), an observer is designed to reconstruct the state in the domain.

With Dirichlet actuation, observer of the following form

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu_x(0,t) + P_0(u_x(0,t) - \hat{u}_x(0,t)) \quad (36)$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + p_1(x)(u_x(0,t) - \hat{u}_x(0,t)), \quad x \in (0,l) \quad (37)$$

$$\hat{u}(0,t) = C\hat{X}(t) + p_2(u_x(0,t) - \hat{u}_x(0,t)) \quad (38)$$

$$\hat{u}(l,t) = U(t) \quad (39)$$

is considered, where the constant vector $P_0$, the function $p_1(x)$, and the constant $p_2$ are to be determined.

Write the observer error as

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \qquad \tilde{X}(t) = X(t) - \hat{X}(t)$$

then, to achieve exponential stability of the observer error system

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}_x(0,t) \quad (40)$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) - p_1(x)\tilde{u}_x(0,t), \quad x \in (0,l) \quad (41)$$

$$\tilde{u}(0,t) = C\tilde{X}(t) - p_2\tilde{u}_x(0,t), \quad \tilde{u}(l,t) = 0 \quad (42)$$

the following transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \Theta(x)\tilde{X}(t) \quad (43)$$

is introduced to transform (40)–(42) into the exponentially stable system

$$\dot{\tilde{X}}(t) = (A - P_0\Theta'(0))\tilde{X}(t) - P_0\tilde{w}_x(0,t) \quad (44)$$

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t), \quad x \in (0,l) \quad (45)$$

$$\tilde{w}(0,t) = 0, \qquad \tilde{w}(l,t) = 0 \quad (46)$$

where $\Theta(x)$ is to be determined, and $P_0$ is chosen such that $A - P_0\Theta'(0)$ is a Hurwitz matrix.

By matching (40)–(42) and (44)–(46), a sufficient condition is obtained:

$$\Theta''(x) - \Theta(x)A = 0 \quad (47)$$

$$\Theta(0) = C, \qquad \Theta(l) = 0 \quad (48)$$

and

$$p_1(x) = \Theta(x)P_0 \quad (49)$$

$$p_2 = 0. \quad (50)$$

So, it is only needed to solve the problem of differential equation (47)–(48). To construct the solution to the ODE (47)–(48), a lemma is shown first.

**Lemma 1.** *Write*

$$F = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}, \qquad G = (0 \quad I)e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

*then $G$ is a nonsingular matrix if and only if the matrix $A$ has no eigenvalues of the form $-k^2\pi^2/l^2$ for $k \in \mathbb{N}$.*

**Proof.** First, there exists an invertible matrix $H$ such that $H^{-1}AH$ is the Jordan's canonical form, that is

$$H^{-1}AH = \mathrm{diag}(J_1 \quad \cdots \quad J_p)$$

where each Jordan block $J_q$, $1 \le q \le p$, is a square matrix of lower-triangular type, and all the elements on its main diagonal are the eigenvalues of $A$, which are denoted by $\varsigma_j, j = 1, 2, \ldots, n$.

Second, a simple calculation gives that

$$G = \sum_{m=0}^{\infty} \frac{l^{2m+1}}{(2m+1)!}A^m.$$

Thus

$$L := H^{-1}GH = \sum_{m=0}^{\infty} \frac{l^{2m+1}}{(2m+1)!}\mathrm{diag}(J_1^m \quad \cdots \quad J_p^m)$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{l(l^2\varsigma_1)^m}{(2m+1)!} & & 0 \\ & \ddots & \\ * & & \sum_{m=0}^{\infty} \frac{l(l^2\varsigma_n)^m}{(2m+1)!} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sinh(l\varsigma_1^{\frac{1}{2}})}{\varsigma_1^{\frac{1}{2}}} & & 0 \\ & \ddots & \\ * & & \frac{\sinh(l\varsigma_n^{\frac{1}{2}})}{\varsigma_n^{\frac{1}{2}}} \end{pmatrix}.$$

Therefore, matrix $L$ is singular if and only if $l\varsigma_j^{1/2} = k\pi i$ for some $\varsigma_j$, $1 \le j \le n$ and $k \in \mathbb{N}$, where $i$ stands for the imaginary unit, namely, the square root of $-1$. Thus, $G$ is a nonsingular matrix if and only if $A$ has no eigenvalues of the form $-k^2\pi^2/l^2$ for $k \in \mathbb{N}$. □

The solution to the Eqs. (47)–(48) can be represented by

$$\Theta(x) = (C \quad \Theta'(0))e^{Fx}\begin{pmatrix} I \\ 0 \end{pmatrix}. \quad (51)$$

Especially, for $x = l$, it holds that

$$(C \quad \Theta'(0))e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix} = \Theta(l) = 0.$$

When $A$ has no eigenvalues of the form $-k^2\pi^2/l^2$ for $k \in \mathbb{N}$, it can be obtained that

$$\Theta'(0) = -C(I \quad 0)e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix}G^{-1}.$$

Thus the explicit solution to the Eqs. (47)–(48) is

$$\Theta(x) = \left(C \quad -C(I \quad 0)e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix}G^{-1}\right)e^{Fx}\begin{pmatrix} I \\ 0 \end{pmatrix}. \quad (52)$$

Choose $P_0$ such that $A - P_0\Theta'(0)$ is Hurwitz, then $p_1(x)$ and $p_2$ are determined through (49) and (50). Thus, all the quantities needed to implement the observer (36)–(39) are determined.

The system (44)–(46) is a cascade of the exponentially stable heat equation (45)–(46) and the exponentially stable ODE (44). The entire observer error system is exponentially stable.

**Theorem 2.** *Assume that the matrix $A$ has no eigenvalues of the form $-k^2\pi^2/l^2$ for $k \in \mathbb{N}$, then the observer (36)–(39) with gains defined through (49), (50) and (52), guarantees that the observer error system is exponentially stable in the sense of the norm*

$$\|(\tilde{X}(t), \tilde{u}(\cdot,t))\|^2 = |\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)}$$

*that is, $\hat{X}(t)$ and $\hat{u}(t)$ exponentially track $X(t)$ and $u(t)$ in the sense of above norm.*

**Proof.** From the transformation (43), the following relations

$$\|\tilde{w}\|^2 \le 2\|\tilde{u}\|^2 + 2\|\Theta\|^2|\tilde{X}|^2, \qquad \|\tilde{w}_x\|^2 \le 2\|\tilde{u}_x\|^2 + 2\|\Theta'\|^2|\tilde{X}|^2$$

$$\|\tilde{u}\|^2 \le 2\|\tilde{w}\|^2 + 2\|\Theta\|^2|\tilde{X}|^2, \qquad \|\tilde{u}_x\|^2 \le 2\|\tilde{w}_x\|^2 + 2\|\Theta'\|^2|\tilde{X}|^2$$

are obtained. With the Lyapunov function

$$\tilde{V}(t) = \tilde{X}^T\tilde{P}\tilde{X} + \frac{\tilde{a}}{2}\|\tilde{w}(\cdot,t)\|^2 + \frac{1}{2}\|\tilde{w}_x(\cdot,t)\|^2$$

where $\tilde{P} = \tilde{P}^T > 0$ is the solution to the Lyapunov equation

$$\tilde{P}(A - P_0\Theta'(0)) + (A - P_0\Theta'(0))^T\tilde{P} = -\tilde{Q}$$

for some $\tilde{Q} = \tilde{Q}^T > 0$, it can be obtained that

$$\underline{\varrho}(|\tilde{X}(t)|^2 + \|\tilde{u}(t)\|^2_{H^1(0,l)}) \le \tilde{V} \le \overline{\varrho}(|\tilde{X}(t)|^2 + \|\tilde{u}(t)\|^2_{H^1(0,l)})$$

where

$$\underline{\varrho} = \frac{\min\left\{\frac{\tilde{a}}{2}, \frac{1}{2}, \lambda_{\min}(\tilde{P})\right\}}{\max\{2, 1 + (\|\Theta'\|^2 + \tilde{a}\|\Theta\|^2)/\lambda_{\min}(\tilde{P})\}}$$

$$\overline{\varrho} = \max\{\tilde{a}, 1, \|\Theta'\|^2 + \tilde{a}\|\Theta\|^2 + \lambda_{\max}(\tilde{P})\}.$$

Calculate the time derivative of the Lyapunov function along the solutions to the system (44)–(46), then

$$\dot{\tilde{V}} \le -\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2 - \left(\tilde{a} - 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} - \frac{1+l}{l}\right)\|\tilde{w}_x\|^2 - \tilde{w}_x(0,t)^2$$

where the last inequality is obtained by Agmon's inequality and the following inequality

$$-\|\tilde{w}_{xx}\|^2 \le \frac{1+l}{l}\|\tilde{w}_x\|^2 - \tilde{w}_x(0,t)^2.$$

Take

$$\tilde{a} > 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} + \frac{1+l}{l}$$

and by Poincaré inequality, then

$$\dot{\tilde{V}} \le -\tilde{b}\tilde{V}$$

where

$$\tilde{b} = \min\left\{\frac{\lambda_{\min}(\tilde{Q})}{2\lambda_{\max}(\tilde{P})}, \frac{2}{1+4l^2}\left(1 - 2\frac{|\tilde{P}P_0|^2}{\tilde{a}\lambda_{\min}(\tilde{Q})} - \frac{1+l}{\tilde{a}l}\right)\right\} > 0.$$

Hence

$$|\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)} \le \varrho(|\tilde{X}(0)|^2 + \|\tilde{u}(\cdot,0)\|^2_{H^1(0,l)})e^{-\tilde{b}t}$$

for all $t \ge 0$ with $\varrho = \overline{\varrho}/\underline{\varrho}$, which means that the error system (40)–(42) is exponentially stable in the sense of the norm

$$\|(\tilde{X}(t), \tilde{u}(\cdot,t))\|^2 = |\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)}$$

and thus completes the proof. □

Replace $u(y,t)$ and $X(t)$ by $\hat{u}(y,t)$ and $\hat{X}(t)$ in (28) respectively, then an output feedback control law is obtained as follows

$$U(t) = \int_0^l \Phi(l-y)\hat{u}(y,t)\mathrm{d}yB + \Phi(l)\hat{X}(t). \tag{53}$$

**Theorem 3.** *Assume that the matrix A has no eigenvalues of the form $-k^2\pi^2/l^2$ for $k \in \mathbb{N}$, then for any initial data $X(0), \hat{X}(0) \in \mathbb{R}$ and $u(\cdot,0), \hat{u}(\cdot,0) \in H^1(0,l)$, the closed-loop system consisting of the plant (1)–(4), the controller (53) and the observer (36)–(39) has a classical solution which is exponentially stabilized in the sense of the norm*

$$\|(X(t), u(\cdot,t), \hat{X}(t), \hat{u}(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)} + |\hat{X}(t)|^2 + \|\hat{u}(\cdot,t)\|^2_{H^1(0,l)}.$$

**Proof.** The transformation

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x \Phi(x-y)\hat{u}(y,t)\mathrm{d}yB - \Phi(x)\hat{X}(t) \tag{54}$$

transforms (36)–(39) into the system

$$\dot{\hat{X}}(t) = (A+BK)\hat{X}(t) + B\hat{w}_x(0,t) + (B+P_0)(\tilde{w}_x(0,t) + \Theta'(0)\tilde{X}(t)) \tag{55}$$

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \left(p_1(x) - \Phi(x)(B+P_0) - \int_0^x \Phi(x-y)p_1(y)\mathrm{d}yB\right) \times (\tilde{w}_x(0,t) + \Theta'(0)\tilde{X}(t)), \quad x \in (0,l) \tag{56}$$

$$\hat{w}(0,t) = 0, \qquad \hat{w}(l,t) = 0. \tag{57}$$

The $(\tilde{X}, \tilde{w})$-system (44)–(46) and the homogeneous part of the $(\hat{X}, \hat{w})$-system (55)–(57) (without $\tilde{X}(t)$, $\tilde{w}(0,t)$) are exponentially stabilized. The interconnection of the two systems $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$ is a cascade. The combined $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$-system is exponentially stabilized. In fact, this fact can be proved through the weighted Lyapunov function

$$E(t) = \hat{X}^T\hat{P}\hat{X} + \frac{\hat{a}}{2}\|\hat{w}(\cdot,t)\|^2 + \frac{1}{2}\|\hat{w}_x(\cdot,t)\|^2 + e\tilde{V}(t) \tag{58}$$

where the matrix $\hat{P} = \hat{P}^T > 0$ is the solution to the Lyapunov equation

$$\hat{P}(A+BK) + (A+BK)^T\hat{P} = -\hat{Q}$$

for some $\hat{Q} = \hat{Q}^T > 0$, the constant $\hat{a}$ and the weighting constant $e$ are to be chosen later.

Calculate the time derivative of (58), then

$$\begin{aligned}\dot{E} \le\ & -\hat{X}^T\hat{Q}\hat{X} + 2\hat{X}^TP(B\hat{w}_x(0,t) + (B+P_0)(\tilde{w}_x(0,t) \\ & + \Theta'(0)\tilde{X}(t))) - \hat{a}\|\hat{w}_x\|^2 + \hat{a}\int_0^l \hat{w}(x) \\ & \times \left(p_1(x) - \Phi(x)(B+P_0) - \int_0^x \Phi(x-y)p_1(y)\mathrm{d}yB\right) \\ & \times (\tilde{w}_x(0,t) + \Theta'(0)\tilde{X}(t))dx - \|\hat{w}_{xx}\|^2 \\ & + \int_0^l \hat{w}_x(x)\left(p_1'(x) - \Phi'(x)(B+P_0) - CBp_1(x)\right. \\ & \left. - \int_0^x \Phi'(x-y)p_1(y)\mathrm{d}yB\right)(\tilde{w}_x(0,t) + \Theta'(0)\tilde{X}(t))dx \\ & + e\left(-\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2 - \left(\tilde{a} - 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} - \frac{1+l}{l}\right)\|\tilde{w}_x\|^2\right).\end{aligned}$$

Let

$$\theta = \max\left\{p_1(x) - \Phi(x)(B+P_0) - \int_0^x \Phi(x-y)p_1(y)\mathrm{d}yB\right\}$$

$$\vartheta = \max\left\{p_1'(x) - \Phi'(x)(B+P_0) - CBp_1(x) - \int_0^x \Phi'(x-y)p_1(y)\mathrm{d}yB\right\}$$

then by Poincaré, Agmon's and Young inequalities and after some complex calculations, it can be obtained that

$$\dot{E} \le -e_1|\hat{X}|^2 - e_2\|\hat{w}_x\|^2 - e_3|\tilde{X}|^2 - e_4\|\tilde{w}_x\|^2$$

where

$$e_1 = \frac{\lambda_{\min}(\hat{Q})}{2} - \epsilon|P(B+P_0)|^2,$$

$$e_2 = \frac{\hat{a}}{2} - \frac{1}{2} - 4\frac{|PB|^2}{\lambda_{\min}(\hat{Q})} - \frac{1+l}{l}$$

$$e_3 = \frac{\lambda_{\min}(\tilde{Q})}{2}e - \left(\frac{1}{\epsilon} + 4\hat{a}\theta^2l^3 + \vartheta^2l\right)|\Theta'(0)|^2$$

$$e_4 = e\left(\tilde{a} - 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} - \frac{1+l}{l}\right) - 4\frac{|P(B+P_0)|^2}{\lambda_{\min}(\hat{Q})} - 4\hat{a}\theta^2l^3 - \vartheta^2l.$$

Choose positive constants $\hat{a}$ and $\epsilon$ such that

$$\hat{a} > 8\frac{|PB|^2}{\lambda_{\min}(\hat{Q})} + \frac{3l+2}{l}, \qquad \epsilon < \frac{\lambda_{\min}(\hat{Q})}{2|P(B+P_0)|^2}$$

further choose a positive constant $e$ to satisfy

$$e > \frac{2}{\lambda_{\min}(\tilde{Q})}\left(\frac{1}{\epsilon} + 4\hat{a}\theta^2l^3 + \vartheta^2l\right)|\Theta'(0)|^2$$

and positive constant $\tilde{a}$ so that

$$\tilde{a} > 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})} + \frac{1+l}{l} + \frac{1}{e}\left(4\frac{|P(B+P_0)|^2}{\lambda_{\min}(\hat{Q})} + 4\hat{a}\theta^2l^3 + \vartheta^2l\right)$$

then through a lengthy calculation, it can be obtained that $\dot{E} \leq -fE$, where

$$f = \min\left\{\frac{e_1}{\lambda_{\max}(\hat{P})}, \frac{2e_2}{\hat{a}(1+4l^2)}, \frac{e_3}{e\lambda_{\max}(\tilde{P})}, \frac{2e_4}{e\tilde{a}(1+4l^2)}\right\}.$$

Hence, the $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$-system is exponentially stabilized.

Since the transformations (43) and (54) are invertible, exponential stabilization of the $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$-system ensures exponential stabilization of the $(\hat{X}, \hat{u}, \tilde{X}, \tilde{u})$-system. This directly implies stabilization of the closed-loop $(X, u, \hat{X}, \hat{u})$-system. □

## 5. Example

As an example, consider the following scalar coupled control system

$$\dot{X}(t) = X(t) + u_x(0,t) \quad (59)$$
$$u_t(x,t) = u_{xx}(x,t) \quad (60)$$
$$u(0,t) = -X(t), \qquad u(1,t) = U(t). \quad (61)$$

The state feedback controller, observer, output feedback controller and solutions to the closed-loop systems are derived.

### 5.1. State feedback controller and solutions

The feedback gain is taken as $K = -2$ such that $A + BK$ is Hurwitz, then

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Phi(x) = -(1 \quad 3)e^{Dx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the backstepping controller can be derived explicitly through (28), which is

$$U(t) = \int_0^1 \Phi(1-y)u(y,t)dy + \Phi(1)X(t). \quad (62)$$

The target system is

$$\dot{X}(t) = -X(t) + w_x(0,t) \quad (63)$$
$$w_t(x,t) = w_{xx}(x,t), \qquad w(0,t) = 0, \qquad w(1,t) = 0. \quad (64)$$

Furthermore, the solution to the system (59)–(61) and (62) is explicitly available. Suppose an initial condition is $u(x,0) = -5x$ and $X(0) = -10$. First, the explicit solution to the heat equation (64) is obtained by (29), where

$$\mu_m = \frac{5m\pi\cos(m\pi)}{m^4\pi^4 + 3m^2\pi^2 + 1}(1 \quad 3)e^D\begin{pmatrix} 4m^2\pi^2 + 9 \\ -m^2\pi^2 - 5 \end{pmatrix} - \frac{5m\pi(m^2\pi^2 - 6)}{m^4\pi^4 + 3m^2\pi^2 + 1} - \frac{5}{m\pi}.$$

Then, the solution to the closed-loop system (59)–(61) and (62) can be obtained explicitly from (31) and (21), which is

$$X(t) = -10e^{-t} + 2\sum_{m=1}^{\infty}\frac{m\pi}{m^2\pi^2 - 1}e^{-m^2\pi^2 t}(e^{(m^2\pi^2-1)t} - 1)\mu_m \quad (65)$$

$$u(x,t) = 10e^{-t}(\cosh(ix) - 2i\sinh(ix)) + 2\sum_{m=1}^{\infty}e^{-m^2\pi^2 t}\mu_m\nu_m \quad (66)$$

where

$$\nu_m = \frac{1}{m^2\pi^2 - 1}((m^2\pi^2 + 1)\sin(m\pi x) + m\pi\cos(m\pi x) - m\pi e^{(m^2\pi^2-1)t}(\cosh(ix) - 2i\sinh(ix))).$$

From (65) and (66), it is evident that $X(t)$ and $u(x,t)$ exponentially converges to zero as $t$ tends to the infinity.

### 5.2. Observer, output feedback and solutions

Here

$$\Theta'(0) = \coth 1, \qquad \Theta(x) = \coth 1 \sinh x - \cosh x.$$

Take $P_0 = 2\tanh 1$, then the backstepping observer is

$$\dot{\hat{X}}(t) = \hat{X}(t) + u_x(0,t) + 2\tanh 1\left(u_x(0,t) - \hat{u}_x(0,t)\right)$$
$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + 2(\sinh x - \tanh 1\cosh x)(u_x(0,t) - \hat{u}_x(0,t))$$
$$\hat{u}(0,t) = -\hat{X}(t), \quad \hat{u}(1,t) = \int_0^1 \phi(1-y)\hat{u}(y,t)dy + \Phi(1)\hat{X}(t).$$

Taking the observer initial condition $\hat{u}(x,0) = 0$, $\hat{X}(0) = 0$ and following the similar steps as seeking for the solution to the closed-loop system in Section 5.1, the explicit solution to the resulting error system can also be obtained as follows

$$\tilde{X}(t) = -10e^{-t} + 4\tanh 1\sum_{m=1}^{\infty}\frac{m\pi}{m^2\pi^2 - 1}e^{-m^2\pi^2 t} \times (1 - e^{(m^2\pi^2-1)t})\tilde{\mu}_m \quad (67)$$

$$\tilde{u}(x,t) = 10e^{-t}(\cosh x - \coth 1\sinh x) + 2\sum_{m=1}^{\infty}e^{-m^2\pi^2 t}\tilde{\mu}_m\tilde{\nu}_m \quad (68)$$

where

$$\tilde{\mu}_m = -10\frac{m\pi}{m^2\pi^2 + 1} + 5\frac{\cos(m\pi)}{m\pi}$$
$$\tilde{\nu}_m = \sin(m\pi x) + 2\frac{m\pi}{m^2\pi^2 - 1}(\tanh 1\cosh x - \sinh x) \times (e^{(m^2\pi^2-1)t} - 1) .$$

From (67) and (68), it is obvious that the error system is exponentially stabilized.

## 6. Comments

Control design of coupled PDE–ODE systems is an original area. There are many open problems to be considered. This paper is just a beginning for studying the coupled PDE–ODE systems with interaction between the ODE and the PDE. Other coupled PDE–ODE systems with interaction between the ODE and the PDE, such as the coupled system consisting of an ODE and a wave equation, are also subjects of the ongoing research.

## References


[1] X. Zhang, E. Zuazua, Control, observation and polynomial decay for a coupled heat-wave system, C. R. Acad. Sci. Paris Ser. I 336 (2003) 823–828.
[2] X. Zhang, E. Zuazua, Polynomial decay and control of a 1-d model for fluid–structure interaction, C. R. Acad. Sci. Paris Ser. I 336 (2003) 745–750.
[3] X. Zhang, E. Zuazua, Polynomial decay and control of a 1-d hyperbolic–parabolic coupled system, J. Differential Equations 204 (2004) 380–438.
[4] M. Krstic, A. Smyshlyaev, Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays, Systems & Control Letters 57 (2008) 750–758.
[5] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems, Birkhauser, 2009.
[6] M. Krstic, Compensating a string PDE in the actuation or in sensing path of an unstable ODE, IEEE Transaction on Automatic Control 54 (2009) 1362–1368.
[7] M. Krstic, Compensating actuator and sensor dynamics governed by diffusion PDEs, Systems & Control Letters 58 (2009) 372–377.
[8] A. Smyshlyaev, M. Krstic, Backstepping observers for a class of parabolic PDEs, Systems & Control Letters 54 (2005) 613–625.
[9] G.A. Susto, M. Krstic, Control of PDE–ODE cascades with Neumann interconnections, Journal of the Franklin Institute 347 (2010) 284–314.