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Zhongcheng Zhou [a] & Shuxia Tang [b]
[a] School of Mathematics and Statistics, Southwest University, Chongqing, P.R. China
[b] Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA

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Zhongcheng Zhou[a]* and Shuxia Tang[b]

[a]*School of Mathematics and Statistics, Southwest University, Chongqing, P.R. China;* [b]*Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA*

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In this article, the backstepping method is employed to stabilize a coupled wave-ODE system with internal anti-damping by means of decoupling them into a stable cascaded wave-ODE system. At the same time, the existence of the kernels in backstepping transformation and inverse transformation is proved by iterative method.

Keywords: coupled wave-ODE; internal anti-damping; stabilization; backstepping

1. Introduction

In this article, we consider the stabilization of a coupled wave-ODE system

$$\begin{cases} X'(t) = AX(t) + Bu_x(0,t), \\ u_{tt}(x,t) - u_{xx}(x,t) = \lambda(x)u_t(x,t) + \beta(x)u(x,t) + CX(t), \\ u(0,t) = 0, \quad u(1,t) = U(t), \quad X(0) = x_0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,1), \end{cases} \tag{1}$$

where $\lambda(\cdot) \in C^2([0,1])$, $\beta(\cdot) \in C([0,1])$, $X(t) \in \mathbb{R}^n$ is the ODE state, and the pair (A,B) is assumed to be stabilisable; $u(x,t) \in \mathbb{R}$ is the state of wave equation, and C is a known suitable matrix; x_0, $u_0(x)$ and $u_1(x)$ are initial data; $U(t)$ is the boundary control. Here, we call $\lambda(x)u_t(x,t)$ be anti-damping term, in fact, basic wave equation is conservative, for stabilising wave equation, we will add some dissipation through distributed or boundary damping terms. $\lambda(x)u_t(x,t)$ is distributed damping terms. If $\lambda(x)$ is negative, the eigenvalues of wave equation $u_{tt}(x,t) - u_{xx}(x,t) = \lambda(x)u_t(x,t)$ with zero Dirichlet boundary located in left half of the complex plane, therefore, it is stable. However, If $\lambda(x)$ is positive, the eigenvalues of it located in right half of the complex plane, it is unstable, in this article, maybe $\lambda(x)$ is not negative, for this reason, we call $\lambda(x)u_t(x,t)$ be anti-damping term.

Coupled system appeared in many practical control systems such as electromagnetic coupling, mechanical coupling and coupled chemical reactions. As for coupled wave-ODE system, it has a strong physical background, for example, it can describe the vertical displacement of the string and the lower rigid body for the model of two rigid bodies connected by a spring and hanging from an elastic string. As for controllability of coupled PDE-ODE system, Weiss and Zhao (2009) and Zhao and Weiss (2011) have discussed it. As for stabilization of coupled system, there are many important tools for constructing explicit stabilising feedback controller, such as control Lyapunov function, damping, homogeneity, averaging, backstepping and forwarding methods. Among these methods, backstepping method displays several advantages in feedback controller design such as simplicity and numerical effectiveness, therefore, it has been widely used to study the stabilization of PDEs (Liu 2003; Smyshlyaev and Krstic 2004, 2005; Krstic, Guo, Balogh, and Smyshlyaev 2008). For cascaded PDE-ODE system, Krstic (2009a,b,c) and Susto and Krstic (2010) discussed the stabilization of them by backstepping method; for coupled heat-ODE system, Tang and Xie (2010, 2011a,b) and Tang, Xie, and Zhou (2011) discussed the stabilization of them through backstepping method. How to design the boundary feedback controllers of coupled wave-ODE system with internal anti-damping is an interesting problem.

If $\lambda(x) = \lambda$ is constant in (1), one can eliminate the anti-damping term by introducing the new variable $v(x,t) = e^{-\lambda t}u(x,t)$ transferring the u-system into v-system, then one can design the controller for v-system that achieves a decay rate larger than λ. However, this idea does not work for spatially varying $\lambda(x)$. The main idea of this article is to use backstepping transformation

$$w(x,t) = h(x)u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y - \int_0^x s(x,y)u_t(y,t)\mathrm{d}y - M(x)X(t) \tag{2}$$

*Corresponding author. Email: zhouzc@amss.ac.cn

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and the state feedback

$$U(t)=\frac{1}{h(1)}\left\{\int_0^1 k(1,y)u(y,t)\mathrm{d}y+\int_0^1 s(1,y)u_t(y,t)\mathrm{d}y + M(1)X(t)\right\}, \tag{3}$$

where the functions $h=h(x)$, $M(x)$, kernels $k=k(x,y)$ and $s=s(x,y)$ are suitably chosen, converting (1) into

$$\begin{cases} X'(t)=(A+BK)X(t)+Bw_x(0,t),\\ w_{tt}(x,t)-w_{xx}(x,t)=-dw_t(x,t)-cw(x,t),\\ w(0,t)=0,\quad w(1,t)=0,\quad X(0)=x_0,\\ w(x,0)=w_0(x),\quad w_t(x,0)=w_1(x),\quad x\in(0,1)\end{cases} \tag{4}$$

with K being chosen such that $A+BK$ be Hurwitz, c and d be positive constant. The emphasis of this article lies in proving the existence and invertibility of transformation (2).

This article is organised as follows. In Section 2, we find the feedback controller and state main theorem. In Section 3, we prove existence of kernels $k(x,y)$, $s(x,y)$ and $M(x)$ in backstepping transformation (2). In Section 4, we prove the invertibility of transformation (2). In Section 5, we give the proof of Theorem 2.1. In Section 6, we give some remarks and unsolved problem.

2. State feedback controller design and main theorem

The transformation $(X(t),u(x,t))\mapsto(X(t),w(x,t))$ is postulated in the form

$$w(x,t)=h(x)u(x,t)-\int_0^x k(x,y)u(y,t)\mathrm{d}y-\int_0^x s(x,y)u_t(y,t)\mathrm{d}y-M(x)X(t),$$

where kernels $k(x,y)$, $s(x,y)$ and $M(x)$ are to be determined later. The inverse transformation $(X(t),w(x,t))\mapsto(X(t),u(x,t))$ will be postulated in a similar way later. After some detailed computation, we obtain

$$\begin{aligned} w_{tt}(x,t)&=h(x)u_{tt}(x,t)-\int_0^x k(x,y)u_{tt}(y,t)\mathrm{d}y\\ &\quad-\int_0^x s(x,y)u_{ttt}(y,t)\mathrm{d}y-M(x)X''(t)\\ &=h(x)CX(t)+h(x)(\lambda(x)u_t(x,t)+\beta(x)u(x,t)+u_{xx}(x,t))\\ &\quad-\int_0^x k(x,y)(u_{yy}(y,t)+\lambda(y)u_t(y,t)\\ &\quad+\beta(y)u(y,t))\mathrm{d}y-\int_0^x k(x,y)CX(t)\mathrm{d}y\\ &\quad-\int_0^x s(x,y)CX'(t)\mathrm{d}y-\int_0^x s(x,y)\\ &\quad\times\Big(u_{yy}+\lambda(y)u_t+\beta(y)u\Big)_t\mathrm{d}y-M(x)X''(t)\\ &=h(x)CX(t)+h(x)(\lambda(x)u_t(x,t)+\beta(x)u(x,t)+u_{xx}(x,t))\\ &\quad-\int_0^x k(x,y)CX(t)\mathrm{d}y-\int_0^x s(x,y)C(AX(t)+Bu_x(0,t))\mathrm{d}y\\ &\quad-\Big\{k(x,x)u_x(x,t)-k(x,0)u_x(0,t)-k_y(x,x)u(x,t)\\ &\quad+\int_0^x k_{yy}(x,y)u(y,t)\mathrm{d}y\Big\}\\ &\quad-\int_0^x k(x,y)\lambda(y)u_t(y,t)\mathrm{d}y-\int_0^x k(x,y)\beta(y)u(y,t)\mathrm{d}y\\ &\quad-\Big\{s(x,x)u_{xt}(x,t)-s(x,0)u_{xt}(0,t)-s_y(x,x)u_t(x,t)\\ &\quad+\int_0^x s_{yy}(x,y)u_t(y,t)\mathrm{d}y\Big\}\\ &\quad-\int_0^x s(x,y)\beta(y)u_t(y,t)\mathrm{d}y-\int_0^x s(x,y)\lambda(y)(CX(t)\\ &\quad+\beta(y)u(y,t)+\lambda(y)u_t(y,t))\mathrm{d}y\\ &\quad-\Big\{\lambda(x)s(x,x)u_x(x,t)-s(x,0)\lambda(0)u_x(0,t)\\ &\quad-(\lambda(y)s(x,y))_y(x,x)u(x,t)\\ &\quad+\int_0^x(\lambda(y)s(x,y))_{yy}(x,y)u\mathrm{d}y\Big\}\\ &\quad-M(x)\Big(A^2X(t)+ABu_x(0,t)+Bu_{xt}(0,t)\Big). \end{aligned} \tag{5}$$

According to (2), we have

$$\begin{aligned} &-w_{xx}(x,t)+dw_t(x,t)+cw(x,t)\\ &=-\Big\{h''(x)u(x,t)+2h'(x)u_x(x,t)+h(x)u_{xx}\\ &\quad-\frac{\mathrm{d}}{\mathrm{d}x}k(x,x)u-k(x,x)u_x\\ &\quad-\int_0^x k_{xx}(x,y)u(y,t)\mathrm{d}y-k_x(x,x)u(x,t)-\frac{\mathrm{d}}{\mathrm{d}x}s(x,x)u_t\\ &\quad-s(x,x)u_{xt}(x,t)\\ &\quad-\int_0^x s_{xx}(x,y)u_t(y,t)\mathrm{d}y-s_x(x,x)u_t(x,t)-M''(x)X(t)\Big\}\\ &\quad+d\Big\{h(x)u_t(x,t)-\int_0^x k(x,y)u_t(y,t)\mathrm{d}y\\ &\quad-M(x)(AX(t)+Bu_x(0,t))\\ &\quad-\int_0^x s(x,y)(CX(t)+\lambda(y)u_t(y,t)+\beta(y)u(y,t))\mathrm{d}y\\ &\quad-s(x,x)u_x(x,t)+s_y(x,x)u(x,t)+s(x,0)u_x(0,t)\\ &\quad-\int_0^x s_{yy}(x,y)u(y,t)\mathrm{d}y\Big\}\\ &\quad+c\Big\{h(x)u(x,t)-\int_0^x k(x,y)u(y,t)\mathrm{d}y\\ &\quad-\int_0^x s(x,y)u_t(y,t)\mathrm{d}y-M(x)X(t)\Big\}. \end{aligned} \tag{6}$$

By $w_{tt}-w_{xx}+dw_t+cw=0$, (5) and (6), after rearranging the terms, we obtain

$$\begin{aligned}0=&\Big(h(x)C-\int_0^x k(x,y)C\,\mathrm{d}y-\int_0^x s(x,y)CA\,\mathrm{d}y\\&-M(x)A^2-\int_0^x s(x,y)\lambda(y)C\mathrm{d}y.\\&-cM(x)-d\int_0^x s(x,y)C\,\mathrm{d}y-dM(x)A+M''(x)\Big)X(t)\\&+\int_0^x u(y,t)\big(k_{xx}-k_{yy}-(\beta(y)+c)k\\&-(\lambda(y)+d)s_{yy}-(\lambda(y)\beta(y).\\&+\lambda''(y)+d\beta(y))s-2\lambda'(y)s_y)\mathrm{d}y\\&+\int_0^x u_t(y,t)\Big(s_{xx}-s_{yy}-(\lambda(y)+d)k-(\lambda^2(y)\\&+d\lambda(y)+\beta(y)+c)s\Big)\mathrm{d}y\\&+\Big\{h(x)\beta(x)+k_y(x,x)+(\lambda(y)s(x,y))_y(x,x)-h''(x)\\&+\frac{\mathrm{d}}{\mathrm{d}x}k(x,x)+k_x(x,x)+\mathrm{d}s_y(x,x)+ch(x)\Big\}u(x,t)\\&+\Big\{\lambda(x)h(x)+s_x(x,x)+\frac{\mathrm{d}}{\mathrm{d}x}s(x,x)\\&+s_y(x,x)+dh(x)\Big\}u_t(x,t)\\&+\Big\{-k(x,x)-\lambda(x)s(x,x)-2h'(x)\\&+k(x,x)-ds(x,x)\Big\}u_x(x,t)\\&+\Big\{k(x,0)-\int_0^x s(x,y)CB\mathrm{d}y+\lambda(0)s(x,0)\\&-M(x)AB-dM(x)B+ds(x,0)\Big\}u_x(0,t)\\&+\Big\{s(x,0)-M(x)B\Big\}u_{xt}(0,t).\end{aligned}\tag{7}$$

Next, we choose $k(x,y)$, $s(x,y)$ and $M(x)$ satisfying coupled PDEs

$$\begin{cases}k_{xx}-k_{yy}-(\beta(y)+c)k\\-(\lambda(y)+d)s_{yy}-(\lambda(y)\beta(y)+\lambda''(y)\\\quad+d\beta(y))s-2\lambda'(y)s_y=0,\\2\dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x)+(\lambda(x)+d)s_y(x,x)+\lambda'(x)s(x,x)\\\quad+(\beta(x)+c)h(x)=h''(x),\\k(x,0)=\displaystyle\int_0^x s(x,y)CB\,\mathrm{d}y-\lambda(0)M(x)B+M(x)AB,\end{cases}\tag{8}$$

$$\begin{cases}s_{xx}-s_{yy}-(\lambda(y)+d)k-\big(\lambda^2(y)+d\lambda(y)+\beta(y)+c\big)s=0,\\2\dfrac{\mathrm{d}}{\mathrm{d}x}s(x,x)=-(\lambda(x)+d)h(x),\\(\lambda(x)+d)s(x,x)=-2h'(x),\\s(x,0)=M(x)B\end{cases}\tag{9}$$

and

$$\begin{aligned}&M''(x)-M(x)(A^2+dA+cI_n)-\int_0^x s(x,y)\lambda(y)C\,\mathrm{d}y\\&-\int_0^x s(x,y)CA\,\mathrm{d}y-\int_0^x k(x,y)C\,\mathrm{d}y+h(x)C\\&-d\int_0^x s(x,y)C\,\mathrm{d}y=0.\end{aligned}\tag{10}$$

Choosing $h(0)=1$, according to (2) and (4), we obtain $M(0)=0$ and $M'(0)=K$. By the second and third equality of (9), we obtain $h'(x)h(x)=s(x,x)\frac{\mathrm{d}}{\mathrm{d}x}s(x,x)$. Integrating it and noticing $h(0)=1$, we can choose $s(x,x)=-\sqrt{h^2(x)-1}$. Hence,

$$\frac{h'(x)}{\sqrt{h^2(x)-1}}=\frac{\lambda(x)+d}{2},$$

which shows $h(x)=\cosh\big(\int_0^x a(\tau)\mathrm{d}\tau\big)$, where $a(x)$ is defined by $a(x):=\frac{\lambda(x)+d}{2}$, thus, we have

$$s(x,x)=-\frac{h'(x)}{a(x)}=-\sinh\bigg(\int_0^x a(\tau)\mathrm{d}\tau\bigg).$$

Next, we will compute $k(x,x)$ explicitly. Let us denote $f(x)=s_y(x,x)$. Integrating the second equality of (8) and substituting $s(x,x)$ by $-\frac{h'(x)}{a(x)}$, we have

$$\begin{aligned}2k(x,x)=h'(x)+\int_0^x\Big[&-2a(\tau)f(\tau)+\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\\&-(\beta(\tau)+c)h(\tau)\Big]\mathrm{d}\tau.\end{aligned}\tag{11}$$

By $\frac{\mathrm{d}}{\mathrm{d}x}s(x,x)=-a(x)h(x)$, we have $s_x(x,x)=-a(x)\times h(x)-f(x)$. By the first equality of (9) and substituting y with x, we have

$$\begin{aligned}s_{xx}(x,x)-s_{yy}(x,x)&=(s_x(x,x)-s_y(x,x))'\\&=2a(x)k(x,x)+(\lambda^2(x)+d\lambda(x)\\&\quad+\beta(x)+c)s(x,x)\\&=(-a(x)h(x)-2f(x))'.\end{aligned}\tag{12}$$

By (11) and (12), we obtain

$$\begin{aligned}&-2f'(x)-a'(x)h(x)-a(x)h'(x)\\&=a(x)\bigg\{h'(x)+\int_0^x\Big[-2a(\tau)f(\tau)+\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\\&\quad-(\beta(\tau)+c)h(\tau)\Big]\mathrm{d}\tau\bigg\}\\&\quad+(\lambda^2(x)+d\lambda(x)+\beta(x)+c)s(x,x).\end{aligned}\tag{13}$$

Simplifying (13), we get

$$\begin{aligned}&-2f'(x)+a(x)\int_0^x 2a(\tau)f(\tau)\mathrm{d}\tau\\&=2a(x)h'(x)+a'(x)h(x)+a(x)\int_0^x\bigg[\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\\&\quad-(\beta(\tau)+c)h(\tau)\Big]\mathrm{d}\tau\\&\quad-\Big(\lambda^2(x)+d\lambda(x)+\beta(x)+c\Big)\sinh\bigg(\int_0^x a(\tau)\mathrm{d}\tau\bigg).\end{aligned}\tag{14}$$

Hence,

$$\begin{aligned}2f'(x)-2a(x)\int_0^x a(\tau)f(\tau)\mathrm{d}\tau&=-2a(x)h'(x)-a'(x)h(x)\\&\quad-a(x)\int_0^x\left[\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c)h(\tau)\right]\mathrm{d}\tau\\&\quad+\left(\lambda^2(x)+d\lambda(x)+\beta(x)+c\right)\sinh\left(\int_0^x a(\tau)\mathrm{d}\tau\right).\end{aligned}\tag{15}$$

By (15), $s(x,0)=M(x)B$, $M(0)=0$ and $M'(0)=K$, we obtain that $f(x)$ satisfies

$$\begin{cases}2f'(x)-2a(x)\int_0^x a(\tau)f(\tau)\mathrm{d}\tau=L(x),\\f(0)=-a(0)-KB,\end{cases}\tag{16}$$

where $L(x)$ is defined by

$$\begin{aligned}L(x):&=-2a(x)h'(x)-a'(x)h(x)\\&\quad-a(x)\int_0^x\left[\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c)h(\tau)\right]\mathrm{d}\tau\\&\quad+\left(\lambda^2(x)+d\lambda(x)+\beta(x)+c\right)\sinh\left(\int_0^x a(\tau)\mathrm{d}\tau\right).\end{aligned}\tag{17}$$

Differentiating (16) in both sides, $f(x)$ satisfy

$$\begin{cases}2a(x)f''(x)-2a'(x)f'(x)-2a^3(x)f(x)=L'(x)a(x)\\\quad-L(x)a'(x),\\f(0)=-a(0)-KB,\quad f'(0)=-\dfrac{a'(0)}{2}.\end{cases}\tag{18}$$

Solving (18), we obtain

$$\begin{aligned}f(x)&=(-a(0)-KB)\cosh\left(\int_0^x a(\tau)\mathrm{d}\tau\right)\\&\quad+\frac{1}{2}\int_0^x L(y)\cosh\left(\int_y^x a(\tau)\mathrm{d}\tau\right)\mathrm{d}y.\end{aligned}\tag{19}$$

Hence,

$$\begin{aligned}k(x,x)=m(x):&=\frac{1}{2}h'(x)+\frac{1}{2}\int_0^x\Bigg[-2a(\tau)\\&\quad\times\left\{(-a(0)-KB)\cosh\left(\int_0^\tau a(s)\mathrm{d}s\right)\right.\\&\quad\left.+\frac{1}{2}\int_0^\tau L(y)\cosh\left(\int_y^\tau a(s)\mathrm{d}s\right)\mathrm{d}y\right\}\\&\quad+\frac{\lambda'(\tau)h'(\tau)}{a(\tau)}-(\beta(\tau)+c)h(\tau)\Bigg]\mathrm{d}\tau.\end{aligned}\tag{20}$$

For simplifying the Equations (8), (9) and (10), we introduce $\rho_i(i=1,2,3,4,5)$ as follows. $\rho_1(y)=\lambda(y)+d$, $\rho_2(y)=\beta(y)+c$, $\rho_3(y)=\beta(y)+\lambda''(y)+d\beta(y)$, $\rho_4(y)=2\lambda'(y)$, $\rho_5(y)=\lambda^2(y)+d\lambda(y)+\beta(y)+c$, we obtain the coupled kernels equations

$$\begin{cases}k_{xx}(x,y)-k_{yy}(x,y)=\rho_1 s_{yy}(x,y)+\rho_2 k(x,y)\\\qquad\qquad\qquad+\rho_3 s(x,y)+\rho_4 s_y(x,y),\\k(x,x)=m(x),\\k(x,0)=\int_0^x s(x,y)CB\,\mathrm{d}y-\lambda(0)M(x)B+M(x)AB,\end{cases}\tag{21}$$

$$\begin{cases}s_{xx}(x,y)-s_{yy}(x,y)=\rho_1 k(x,y)+\rho_5 s(x,y),\\s(x,x)=-\sinh\left(\int_0^x a(\tau)\mathrm{d}\tau\right),\\s(x,0)=M(x)B\end{cases}\tag{22}$$

and

$$\begin{cases}M''(x)-M(x)(A^2+dA+cI_n)-\int_0^x s(x,y)\lambda(y)C\,\mathrm{d}y\\\quad-\int_0^x s(x,y)CA\mathrm{d}y-\int_0^x k(x,y)C\,\mathrm{d}y+h(x)C\\\quad-d\int_0^x s(x,y)C\,\mathrm{d}y=0,\\M(0)=0,\\M'(0)=K.\end{cases}\tag{23}$$

Introducing the space $H_L^1(0,1)$ defined by

$$H_L^1(0,1):=\{w\in H^1(0,1)|w(0)=0\}$$

and endowed with the H^1-norm, and denote domain

$$\mathbb{T}:=\{(x,y)\in\mathbb{R}^2|0\le x\le 1,0\le y\le x\}.$$

We state the following main theorem.

Theorem 2.1: *Let* $\lambda(\cdot)\in C^2([0,1])$, $\beta(\cdot)\in C([0,1])$, *there exist functions* $h(\cdot)\in C^2([0,1])$, $M(\cdot)\in C^2([0,1])$, $k(\cdot,\cdot)$ *and* $s(\cdot,\cdot)\in C^2(\mathbb{T})$, *such that for any* $(x_0,u^0(\cdot),u^1(\cdot))\in\mathbb{R}^n\times H_L^1(0,1)\times L^2(0,1)$ *satisfying boundary compatibility conditions*

$$u_0(1)=\frac{1}{h(1)}\left(\int_0^1 k(1,y)u_0(y)dy+\int_0^1 s(1,y)u_1(y)dy+M(1)x_0\right).$$

there exists unique classical solutions $(u(\cdot,\cdot),X(\cdot))$ *of the closed-loop system* (1) *and* (3) *in the space* $C([0,+\infty);H_L^1(0,1))\cap C^1([0,+\infty);L^2(0,1))\times C^1([0,+\infty))$. *Moreover, the closed-loop system is exponentially stable in the sense of the norm*

$$\left(\int_0^1 u^2(x,t)\mathrm{d}x+\int_0^1 u_x^2(x,t)\mathrm{d}x+\int_0^1 u_t^2(x,t)\mathrm{d}x+\|X(t)\|^2\right)^{\frac{1}{2}}.$$

Remark 1: In system (1), if we substitute C with smooth function $C(x)$, the closed-loop system (1) and (3) is also exponentially stable, which can be proved by similar proof procedure of Theorem 2.1 without any difficulty. Here, it should be pointed out that the kernels functions $k(x,y)$, $s(x,y)$ and $M(x)$ depend on $C(x)$, controller (3) is also dependent on $C(x)$.

3. Existence of the kernels

To prove the existence of solution for Equations (21), (22) and (23), we use change of variable

$$\xi = x + y, \quad \eta = x - y.$$

Let us define the functions $G = G(\xi, \eta)$, $G^s = G^s(\xi, \eta)$ by

$$G(\xi,\eta) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right), \quad G^s(\xi,\eta) = s\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right).$$

Setting $g_1(\xi) := m(\frac{\xi}{2})$, $g_2(\xi) := -\sinh(\int_0^{\frac{\xi}{2}} a(\tau)\mathrm{d}\tau)$, $b_i(\xi,\eta) := \rho_i(\frac{\xi-\eta}{2})(i = 1,2,3,4,5)$, $f_1(\xi) = \int_0^{\xi} s(\xi, y)CB\,\mathrm{d}y - \lambda(0)\times M(\xi)B + M(\xi)AB$ and $f_2(\xi) = M(\xi)B$, by (21) and (22), we obtain the coupled PDEs

$$\begin{cases} G_{\xi\eta}(\xi,\eta) = b_1(\xi,\eta)(G^s_{\xi\xi} - 2G^s_{\xi\eta} + G^s_{\eta\eta}) + b_2(\xi,\eta)G \\ \qquad\qquad + b_3(\xi,\eta)G^s + b_4(\xi,\eta)(G^s_\xi - G^s_\eta), \\ G(\xi,0) = g_1(\xi), \\ G(\xi,\xi) = f_1(\xi) \end{cases} \tag{24}$$

and

$$\begin{cases} G^s_{\xi\eta}(\xi,\eta) = b_1(\xi,\eta)G(\xi,\eta) + b_5(\xi,\eta)G^s(\xi,\eta), \\ G^s(\xi,0) = g_2(\xi), \\ G^s(\xi,\xi) = f_2(\xi). \end{cases} \tag{25}$$

Integrating (24), first with respect to η from 0 to η, then with respect to ξ from η to ξ, we obtain

$$\begin{aligned} G(\xi,\eta) &= g_1(\xi) + f_1(\eta) - g_1(\eta) \\ &\quad + \int_\eta^\xi \int_0^\eta b_1(\tau,s)(G^s_{\xi\xi} - 2G^s_{\xi\eta} + G^s_{\eta\eta})\mathrm{d}s\,\mathrm{d}\tau \\ &\quad + \int_\eta^\xi \int_0^\eta b_2(\tau,s)G\,\mathrm{d}s\,\mathrm{d}\tau + \int_\eta^\xi \int_0^\eta [b_3(\tau,s)G^s \\ &\quad + b_4(\tau,s)(G^s_\xi - G^s_\eta)]\mathrm{d}s\,\mathrm{d}\tau. \end{aligned} \tag{26}$$

Similarly, integrating (25), first with respect to η from 0 to η, then with respect to ξ from η to ξ, we get

$$\begin{aligned} G^s(\xi,\eta) &= g_2(\xi) + f_2(\eta) - g_2(\eta) + \int_\eta^\xi \int_0^\eta [b_1(\tau,s)G(\tau,s) \\ &\quad + b_5(\tau,s)G^s(\tau,s)]\mathrm{d}s\,\mathrm{d}\tau. \end{aligned} \tag{27}$$

According to (23), define

$$\begin{aligned} F(x) &:= \int_0^x s(x,y)\lambda(y)C\,\mathrm{d}y + \int_0^x s(x,y)CA\,\mathrm{d}y \\ &\quad + \int_0^x k(x,y)C\,\mathrm{d}y + d\int_0^x s(x,y)C\,\mathrm{d}y, \end{aligned}$$

we have

$$\begin{cases} M''(x) = M(x)(A^2 + dA + cI_n) + F(x) - h(x)C, \\ M(0) = 0, \\ M'(0) = K, \end{cases} \tag{28}$$

then,

$$\begin{cases} Y'(x) = Y(x)L(x) + (0, \tilde{F}(x)), \\ Y(0) = Y_0 := (0, K), \end{cases} \tag{29}$$

where

$$L(x) = \begin{pmatrix} 0 & (A^2 + dA + cI_n) \\ I_n & 0 \end{pmatrix},$$

$Y = (M(x),\ M'(x))$ and $\tilde{F}(x) := F(x) - h(x)C$. Set $\Phi(x)$ be fundamental solution matrix for homogeneous equation $\Phi'(x) = \Phi(x)L(x)$, then,

$$\begin{aligned} Y(x) &= (M(x), M'(x)) = (0,K)\Phi(x) \\ &\quad + \int_0^x (0, \tilde{F}(\tau))\Phi(x)\Phi^{-1}(\tau)\mathrm{d}\tau, \\ &= (0,K)\begin{pmatrix} \Phi_{11}(x) & \Phi_{12}(x) \\ \Phi_{21}(x) & \Phi_{22}(x) \end{pmatrix} \\ &\quad + \int_0^x (0, \tilde{F}(\tau))\begin{pmatrix} \Psi_{11}(x,\tau) & \Psi_{12}(x,\tau) \\ \Psi_{21}(x,\tau) & \Psi_{22}(x,\tau) \end{pmatrix}\mathrm{d}\tau, \end{aligned} \tag{30}$$

where we used the fact that the determinant of matrix $\Phi(x)$ and $\Phi^{-1}(x)$ is non zero and every element of matrix $\Phi(x)$ is continuous and differentiable, at the same time, we divided matrix $\Phi(x)$ and $\Phi(x)\Phi^{-1}(\tau)$ into appropriate block matrix. Hence, we obtain

$$M'(x) = K\Phi_{22}(x) + \int_0^x \tilde{F}(\tau)\Psi_{22}(x,\tau)\mathrm{d}\tau.$$

Thus,

$$M(x) = \int_0^x K\Phi_{22}(y)\mathrm{d}y + \int_0^x \int_0^y \tilde{F}(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y. \tag{31}$$

Substituting $M(x)$ into (26) and (27), we obtain

$$\begin{aligned} G(\xi,\eta) &= g_1(\xi) + \int_0^\eta s(\eta,y)CB\,\mathrm{d}y - \lambda(0)M(\eta)B \\ &\quad + M(\eta)AB - g_1(\eta) \\ &\quad + \int_\eta^\xi \int_0^\eta b_2(\tau,s)G(\tau,s)\mathrm{d}s\,\mathrm{d}\tau \\ &\quad + \int_\eta^\xi \int_0^\eta b_1(\tau,s)(G^s_{\xi\xi} - 2G^s_{\xi\eta} + G^s_{\eta\eta})\mathrm{d}s\,\mathrm{d}\tau \\ &\quad + \int_\eta^\xi \int_0^\eta [b_3(\tau,s)G^{s,n} + b_4(\tau,s)(G^s_\xi - G^s_\eta)]\mathrm{d}s\,\mathrm{d}\tau \\ &= g_1(\xi) + \int_0^\eta s(\eta,y)CB\,\mathrm{d}y \\ &\quad - \lambda(0)\Big\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y + \int_0^\eta \int_0^y (F(\tau) \\ &\quad - h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\Big\}B \end{aligned}$$

$$
\begin{aligned}
&+\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y+\int_0^\eta\int_0^y(F(\tau)\right.\\
&\left.-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}AB-g_1(\eta)\\
&+\int_\eta^\xi\int_0^\eta b_2(\tau,s)G(\tau,s)\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta b_1(\tau,s)(G^s_{\xi\xi}-2G^s_{\xi\eta}+G^s_{\eta\eta})\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta[b_3(\tau,s)G^s+b_4(\tau,s)(G^s_\xi-G^s_\eta)]\mathrm{d}s\,\mathrm{d}\tau\\
&=g_1(\xi)-\lambda(0)\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y\right.\\
&+\int_0^\eta\int_0^y(-h(\tau))C\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\Big\}B\\
&+\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y+\int_0^\eta\int_0^y(-h(\tau)C)\right.\\
&\left.\times\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}AB-g_1(\eta)\\
&+\int_0^\eta s(\eta,y)CB\,\mathrm{d}y\\
&-\lambda(0)\left\{\int_0^\eta\int_0^y F(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B\\
&+\left\{\int_0^\eta\int_0^y F(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}AB\\
&+\int_\eta^\xi\int_0^\eta b_2(\tau,s)G(\tau,s)\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta b_1(\tau,s)(G^s_{\xi\xi}-2G^s_{\xi\eta}+G^s_{\eta\eta})\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta[b_3G^s+b_4(G^s_\xi-G^s_\eta)]\mathrm{d}s\,\mathrm{d}\tau \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
G^s(\xi,\eta)&=g_2(\xi)+M(\eta)B-g_2(\eta)\\
&\quad+\int_\eta^\xi\int_0^\eta[b_1(\tau,s)G(\tau,s)+b_5(\tau,s)G^s(\tau,s)]\mathrm{d}s\,\mathrm{d}\tau\\
&=g_2(\xi)+\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y+\int_0^\eta\int_0^y(F(\tau)\right.\\
&\quad\left.-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B\\
&\quad-g_2(\eta)+\int_\eta^\xi\int_0^\eta[b_1G(\tau,s)+b_5G^s(\tau,s)]\mathrm{d}s\,\mathrm{d}\tau\\
&=g_2(\xi)+\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y\right.\\
&\quad\left.+\int_0^\eta\int_0^y(-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B-g_2(\eta)\\
&\quad+\left\{\int_0^\eta\int_0^y F(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B\\
&\quad+\int_\eta^\xi\int_0^\eta[b_1G(\tau,s)+b_5G^s(\tau,s)]\mathrm{d}s\,\mathrm{d}\tau. \qquad (33)
\end{aligned}
$$

We use a classical iterative method to prove coupled integral equations (32) and (33) have a unique classical solution. We define $G^0(\xi,\eta)$ and $G^{s,0}(\xi,\eta)$ as

$$
\begin{aligned}
G^0(\xi,\eta):=&g_1(\xi)-\lambda(0)\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y\right.\\
&\left.+\int_0^\eta\int_0^y(-h(\tau))C\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B\\
&+\left\{\int_0^\eta K\Phi_{22}(y)\mathrm{d}y\right.\\
&\left.+\int_0^\eta\int_0^y(-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}AB-g_1(\eta),
\end{aligned}
$$

$$
\begin{aligned}
G^{s,0}(\xi,\eta):=&\,g_2(\xi)+\left\{\int_0^\eta K\Phi_{22}(y)\,\mathrm{d}y\right.\\
&\left.+\int_0^\eta\int_0^y(-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B-g_2(\eta).
\end{aligned}
$$

We next construct the following recursion for $n=0,1,2,3,\ldots$

$$
\begin{aligned}
G^{n+1}(\xi,\eta)=&\int_0^\eta G^{s,n}(\eta+y,\eta-y)CB\mathrm{d}y\\
&-\lambda(0)\left\{\int_0^\eta\int_0^y F^n(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}B\\
&+\left\{\int_0^\eta\int_0^y F^n(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right\}AB\\
&+\int_\eta^\xi\int_0^\eta b_2(\tau,s)G^n(\tau,s)\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta b_1(\tau,s)(G^{s,n}_{\xi\xi}-2G^{s,n}_{\xi\eta}+G^{s,n}_{\eta\eta})\mathrm{d}s\,\mathrm{d}\tau\\
&+\int_\eta^\xi\int_0^\eta[b_3(\tau,s)G^{s,n}+b_4(\tau,s)(G^{s,n}_\xi-G^{s,n}_\eta)]\mathrm{d}s\,\mathrm{d}\tau
\end{aligned}
\qquad (34)
$$

and

$$
\begin{aligned}
G^{s,n+1}(\xi,\eta)=&\left(\int_0^\eta\int_0^y F^n(\tau)\Psi_{22}(y,\tau)\mathrm{d}\tau\,\mathrm{d}y\right)B\\
&+\int_\eta^\xi\int_0^\eta[b_1G^n+b_5G^{s,n}]\mathrm{d}s\,\mathrm{d}\tau, \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
F^n(\tau)=&\int_0^\tau G^{s,n}(\tau+y,\tau-y)\lambda(y)C\,\mathrm{d}y\\
&+\int_0^\tau G^{s,n}(\tau+y,\tau-y)CA\,\mathrm{d}y\\
&+\int_0^\tau G^n(\tau+y,\tau-y)C\,\mathrm{d}y\\
&+d\int_0^\tau G^{s,n}(\tau+y,\tau-y)C\,\mathrm{d}y.
\end{aligned}
$$

By the definition of $G^0(\xi,\eta)$ and $G^{s,0}(\xi,\eta)$, we choose number M large enough such that

$$\begin{aligned}
&|G^{s,0}(\xi,\eta)| \le \|g_2'\|_{L^\infty}|\xi+\eta| + \left\|K\Phi_{22}(y) + \int_0^y(-h(\tau)C)\Psi_{22}(y,\tau)\mathrm{d}\tau\right\|_{L^\infty}\|B\||\eta| \\
&\qquad \le M|\xi+\eta|, \\
&|G^0(\xi,\eta)| \le M, \\
&|G^0_\xi(\xi,\eta)| = |g_1'(\xi)| \le \|g_1'\|_{L^\infty} \le \|g_1'\|_{L^\infty}(\xi+\eta)(\xi+\eta)^{-1} \\
&\qquad\qquad \le 4\|g_1'\|_{L^\infty}(\xi+\eta)^{-1} \le M(\xi+\eta)^{-1}, \\
&|G^0_\eta(\xi,\eta)| \le M(\xi+\eta)^{-1}, \\
&|G^{s,0}_\xi(\xi,\eta)|, |G^{s,0}_\eta(\xi,\eta)| \le M, \\
&|G^{s,0}_{\xi\xi}(\xi,\eta)|, |G^{s,0}_{\xi\eta}(\xi,\eta)|, |G^{s,0}_{\eta\eta}(\xi,\eta)| \le M(\xi+\eta)^{-1}.
\end{aligned}$$

Supposing that for some $n\in\mathbb{N}$, we have

$$\begin{aligned}
&|G^n(\xi,\eta)| \le MN^n\frac{(\xi+\eta)^n}{n!}, \\
&|G^{s,n}(\xi,\eta)| \le MN^n\frac{(\xi+\eta)^{n+1}}{(n+1)!}, \\
&|G^n_\xi(\xi,\eta)|, |G^n_\eta(\xi,\eta)| \le MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}, \\
&|G^{s,n}_\xi(\xi,\eta)|, |G^{s,n}_\eta(\xi,\eta)| \le MN^n\frac{(\xi+\eta)^n}{n!}, \\
&|G^{s,n}_{\xi\xi}(\xi,\eta)|, |G^{s,n}_{\xi\eta}(\xi,\eta)|, |G^{s,n}_{\eta\eta}(\xi,\eta)| \le MN^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}.
\end{aligned} \tag{36}$$

where N chosen later. Next, we prove that (36) holds for $n+1$. By the expression of $F^n(\tau)$ and (36), there exists a constant μ large enough which change line to line, such that

$$\begin{aligned}
|F^n(\tau)| &\le MN^n\frac{(2\tau)^{n+1}}{(n+1)!}(\|\lambda(\cdot)C\|_{L^\infty}+\|CA\|)\tau \\
&\quad + MN^n\frac{(2\tau)^n}{n!}(\|d(\cdot)\|_{L^\infty}\|C\|+\|C\|)\tau \\
&\le MN^n\frac{(2\tau)^n}{n!}\Big(\|\lambda(\cdot)C\|_{L^\infty}+\|CA\| \\
&\quad + \|d(\cdot)\|_{L^\infty}\|C\|+\|C\|\Big)\tau \\
&\le \mu MN^n\frac{(2\tau)^n}{n!}.
\end{aligned} \tag{37}$$

According to (35), we have

$$\begin{aligned}
|G^{s,n+1}(\xi,\eta)| &\le \int_0^\eta\int_0^y \mu MN^n\frac{(2\tau)^n}{n!}\mathrm{d}\tau\,\mathrm{d}y \\
&\quad + \|b_1\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \|b_5\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^{s,n}(\tau,s)|\mathrm{d}s\,\mathrm{d}\tau \\
&\le \int_0^\eta\int_0^y \mu MN^n\frac{(2\tau)^n}{n!}\mathrm{d}\tau\,\mathrm{d}y \\
&\quad + \mu MN^n\int_\eta^\xi\int_0^\eta\frac{(\tau+s)^n}{n!}\mathrm{d}s\,\mathrm{d}\tau \\
&\le \mu MN^n\frac{(2\eta)^{n+2}}{(n+2)!} + \mu\frac{MN^n}{(n+1)!} \\
&\quad \times\int_\eta^\xi\big((\tau+\eta)^{n+1}-\tau^{n+1}\big)\mathrm{d}\tau \\
&\le \mu MN^n\frac{(\xi+\eta)^{n+2}}{(n+2)!} + \mu MN^n\frac{(\xi+\eta)^{n+2}}{(n+2)!} \\
&\le MN^{n+1}\frac{(\xi+\eta)^{n+2}}{(n+2)!},
\end{aligned} \tag{38}$$

where N chosen large enough. Next, we estimate $|G^{n+1}(\xi,\eta)|$. By (34) and (36), we obtain

$$\begin{aligned}
\int_0^\eta G^{s,n}(\eta+y,\eta-y)CB\,\mathrm{d}y &\le \mu MN^n\frac{(2\eta)^{n+1}}{(n+1)!} \\
&\le \mu MN^n\frac{(\xi+\eta)^{n+1}}{(n+1)!},
\end{aligned}$$

$$\begin{aligned}
|G^{n+1}(\xi,\eta)| &\le \mu MN^n\frac{(\xi+\eta)^{n+1}}{(n+1)!} + \int_0^\eta\int_0^y 2\mu MN^n\frac{(2\tau)^n}{n!}\mathrm{d}\tau\,\mathrm{d}y \\
&\quad + \|b_2\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \|b_3\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^{s,n}(\tau,s)|\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \|b_1\|_{L^\infty}\int_\eta^\xi\int_0^\eta (|G^{s,n}_{\xi\xi}|+2|G^{s,n}_{\xi\eta}|+|G^{s,n}_{\eta\eta}|)\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \|b_4\|_{L^\infty}\int_\eta^\xi\int_0^\eta (|G^{s,n}_\xi|+|G^{s,n}_\eta|)\mathrm{d}s\,\mathrm{d}\tau \\
&\le \mu MN^n\frac{(\xi+\eta)^{n+1}}{(n+1)!} + \mu MN^n\int_\eta^\xi\int_0^\eta\frac{(\tau+s)^n}{n!}\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \mu MN^n\int_\eta^\xi\int_0^\eta\frac{(\tau+s)^{n-1}}{(n-1)!}\mathrm{d}s\,\mathrm{d}\tau \\
&\le MN^{n+1}\frac{(\xi+\eta)^{n+1}}{(n+1)!},
\end{aligned} \tag{39}$$

where N chosen large enough and greater than μ. In a very similar way, we obtain

$$\begin{aligned}
&|G^{n+1}_\xi(\xi,\eta)|,\ |G^{n+1}_\eta(\xi,\eta)| \le MN^{n+1}\frac{(\xi+\eta)^n}{n!}, \\
&|G^{s,n+1}_\xi(\xi,\eta)|,\ |G^{s,n+1}_\eta(\xi,\eta)| \le MN^{n+1}\frac{(\xi+\eta)^{n+1}}{(n+1)!}, \\
&|G^{s,n+1}_{\xi\xi}(\xi,\eta)|,\ |G^{s,n+1}_{\xi\eta}(\xi,\eta)|,\ |G^{s,n+1}_{\eta\eta}(\xi,\eta)| \le MN^{n+1}\frac{(\xi+\eta)^n}{n!}.
\end{aligned} \tag{40}$$

Thus, by induction we have proved that (36) holds with N large enough only depending on the norm of some

known functions. Once the estimates (36) are proved, it follows that the solutions (26) and (27) are given by the series

$$G^s(\xi,\eta)=\sum_{n=0}^{\infty}G^{s,n}(\xi,\eta),\quad G(\xi,\eta)=\sum_{n=0}^{\infty}G^{n}(\xi,\eta),$$

which are two continuous functions. By the fact $b_i(\cdot,\cdot)(i=1,2,3,4,5)$ be continuous functions, we obtain that $G(\cdot,\cdot)$, $G^s(\cdot,\cdot)$ and $M(\cdot)$ are C^2 functions in terms of (24), (25) and (31). Hence, we obtain the following existence theorem of kernels $k(\cdot,\cdot)$, $s(\cdot,\cdot)$ and $M(\cdot)$.

Theorem 3.1: *Let* $\lambda(\cdot)\in C^2([0,1])$, $\beta(\cdot)\in C([0,1])$, $c>0$, $d>0$, *hence*, *Equations* (21), (22) *and* (23) *have unique classical solutions* $M(\cdot)\in C^2([0,1])$, $k(\cdot,\cdot)$ *and* $s(\cdot,\cdot)\in C^2(\mathbb{T})$.

4. Inverse transformation

Next, we show transformation (2) is invertible and inverse transformation defined as

$$u(x,t)=g(x)w(x,t)+\int_0^x l(x,y)w(y,t)\mathrm{d}y+\int_0^x n(x,y)w_t(y,t)\mathrm{d}y+N(x)X(t),\quad(41)$$

where kernels $l(x,y)$, $n(x,y)$, $g(x)$ and $N(x)$ will be chosen followed the similar procedure in Section 3. According to (4) and (41), we have

$$\begin{aligned}
&u_{tt}(x,t)\\
&=g(x)w_{tt}(x,t)+\int_0^x l(x,y)w_{tt}(y,t)\mathrm{d}y\\
&\quad+\int_0^x n(x,y)w_{ttt}(y,t)\mathrm{d}y+N(x)X''(t)\\
&=g(x)(-dw_t(x,t)-cw(x,t)+w_{xx}(x,t))\\
&\quad+\Big\{l(x,x)w_x(x,t)-l(x,0)w_x(0,t)-l_y(x,x)w(x,t)\\
&\quad+\int_0^x l_{yy}(x,y)w(y,t)\mathrm{d}y\Big\}\\
&\quad-d\int_0^x l(x,y)w_t(y,t)\mathrm{d}y-c\int_0^x l(x,y)w(y,t)\mathrm{d}y\\
&\quad+\Big\{n(x,x)w_{xt}(x,t)-n(x,0)w_{xt}(0,t)-n_y(x,x)w_t(x,t)\\
&\quad+\int_0^x n_{yy}(x,y)w_t(y,t)\mathrm{d}y\Big\}\\
&\quad-c\int_0^x n(x,y)w_t(y,t)\mathrm{d}y-d\int_0^x n(x,y)(-cw(y,t)\\
&\quad-dw_t(y,t))\mathrm{d}y\\
&\quad-d\Big\{n(x,x)w_x(x,t)-n(x,0)w_x(0,t)-n_y(x,x)w(x,t)\\
&\quad+\int_0^x n_{yy}(x,y)w(y,t)\mathrm{d}y\Big\}\\
&\quad+N(x)(A+BK)^2X(t)+N(x)(A+BK)Bw_x(0,t)\\
&\quad+N(x)Bw_{xt}(0,t).
\end{aligned}\tag{42}$$

According to (41), we have

$$\begin{aligned}
&-(u_{xx}+\lambda(x)u_t+\beta(x)u+CX(t))\\
&=-\Big\{g''(x)w+2g'(x)w_x+g(x)w_{xx}\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}x}l(x,x)w+l(x,x)w_x\\
&\quad+\int_0^x l_{xx}(x,y)w(y,t)\mathrm{d}y+l_x(x,x)w(x,t)\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}x}n(x,x)w_t+n(x,x)w_{xt}(x,t)\\
&\quad+\int_0^x n_{xx}(x,y)w_t(y,t)\mathrm{d}y+n_x(x,x)w_t(x,t)+N''(x)X(t)\Big\}\\
&\quad-\lambda(x)\Big\{g(x)w_t(x,t)+\int_0^x l(x,y)w_t(y,t)\mathrm{d}y\\
&\quad+N(x)((A+BK)X(t)+Bw_x(0,t))\\
&\quad+\int_0^x n(x,y)(-dw_t(y,t)-cw(y,t))\mathrm{d}y\\
&\quad+n(x,x)w_x(x,t)-n_y(x,x)w(x,t)\\
&\quad-n(x,0)w_x(0,t)+\int_0^x n_{yy}(x,y)w(y,t)\mathrm{d}y\Big\}\\
&\quad-\beta(x)\Big\{g(x)w(x,t)+\int_0^x l(x,y)w(y,t)\mathrm{d}y\\
&\quad+\int_0^x n(x,y)w_t(y,t)\mathrm{d}y+N(x)X(t)\Big\}-CX(t).
\end{aligned}\tag{43}$$

By $u_{tt}-u_{xx}-\lambda(x)u_t-\beta(x)u-CX(t)=0$, (42) and (43), after rearranging the terms, we obtain

$$\begin{aligned}
0=&\Big(N(x)(A+BK)^2-N''(x)-\lambda(x)N(x)(A+BK)\\
&-\beta(x)N(x)-C\Big)X(t)\\
&+\int_0^x w(y,t)\Big\{-l_{xx}+l_{yy}-cl(x,y)\\
&+cdn(x,y)+dn_{yy}(x,y)\\
&+c\lambda(x)n(x,y)-\lambda(x)n_{yy}(x,y)-\beta(x)l(x,y)\Big\}\mathrm{d}y\\
&+\int_0^x w_t(y,t)\Big\{-n_{xx}+n_{yy}-dl(x,y)-cn(x,y)\\
&+d^2n(x,y)-\lambda(x)l(x,y)\\
&+d\lambda(x)n(x,y)-\beta(x)n(x,y)\Big\}\mathrm{d}y\\
&+\Big\{-cg(x)-l_y(x,x)+dn_y(x,x)\\
&-g''(x)-\frac{\mathrm{d}}{\mathrm{d}x}l(x,x)-l_x(x,x)\\
&+\lambda(x)n_y(x,x)-\beta(x)g(x)\Big\}w(x,t)\\
&+\Big\{-dg(x)-n_y(x,x)-\frac{\mathrm{d}}{\mathrm{d}x}n(x,x)\\
&-n_x(x,x)-\lambda(x)g(x)\Big\}w_t(x,t)
\end{aligned}$$

$$+\Big\{l(x,x)-dn(x,x)-2g'(x)-l(x,x)-\lambda(x)n(x,x)\Big\}w_x(x,t)+\Big\{dn(x,0)+N(x)(A+BK)B-\lambda(x)N(x)B+\lambda(x)n(x,0)-l(x,0)\Big\}w_x(0,t)+\Big\{N(x)B-n(x,0)\Big\}w_{xt}(0,t). \quad (44)$$

Next, we choose $l(x,y)$, $n(x,y)$ and $N(x)$ to satisfy coupled PDEs

$$\begin{cases} l_{xx}(x,y)-l_{yy}(x,y)=-(\beta(x)+c)l(x,y)+c(\lambda(x)+d)n(x,y)+(-\lambda(x)+d)n_{yy}(x,y)\\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}l(x,x)=-g''(x)-(\beta(x)+c)\,g(x)+(\lambda(x)+d)n_y(x,x),\\ l(x,0)=N(x)Bd+N(x)(A+BK)B, \end{cases} \quad (45)$$

$$\begin{cases} n_{xx}(x,y)-n_{yy}(x,y)=-(\lambda(x)+d)l(x,y)+(d\lambda(x)-\beta(x)+d^2-c)n(x,y),\\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}n(x,x)=-(\lambda(x)+d)\,g(x),\\ (\lambda(x)+d)n(x,x)=-2g'(x),\\ n(x,0)=N(x)B \end{cases} \quad (46)$$

and

$$N''(x)=N(x)(A+BK)^2-\lambda(x)N(x)(A+BK)-\beta(x)N(x)-C. \quad (47)$$

By the similar procedure in Section 3, Choosing $g(0)=1$, according to (1) and (41), we obtain $N(0)=0$ and $N'(0)=K$. Then, $N(x)$ satisfy

$$\begin{cases} N''(x)=N(x)(A+BK)^2-\lambda(x)N(x)(A+BK)-\beta(x)N(x)-C,\\ N(0)=0,\\ N'(0)=K. \end{cases} \quad (48)$$

Obviously, Equation (48) have a unique classical solution. By the second and third equation of (46), we obtain $g(x)=\cosh\big(\int_0^x b(\tau)\mathrm{d}\tau\big)$, where $b(x)$ is defined by $b(x):=\frac{\lambda(x)+d}{2}$. Because Equations (45) and (46) are very similar with Equations (21) and (22), by the similar proof procedure, we obtain the following theorem which shows the existence of the kernels $l(\cdot,\cdot)$, $n(\cdot,\cdot)$ and $N(\cdot)$ in inverse transformation (41).

Theorem 4.1: *Let* $\lambda(\cdot)\in C^2([0,1])$, $\beta(\cdot)\in C([0,1])$, $c>0$, $d>0$, *hence*, *Equations* (45), (46) *and* (48) *have unique classical solutions* $N(\cdot)\in C^2([0,1])$, $l(\cdot,\cdot)$ *and* $n(\cdot,\cdot)\in C^2(\mathbb{T})$.

5. Proof of Theorem 2.1

We give the exponential stability of object system (4) in the following lemma.

Lemma 5.1: *For* $c>0$, $d>0$, $A+BK$ *be Hurwitz*, *there exists* ω, $\gamma>0$ *such that for any* $(w_0(\cdot),w_1(\cdot),x_0)\in H_0^1(0,1)\times L^2(0,1)\times\mathbb{R}^n$, *the solution of* (4) *satisfies*

$$\|(w(\cdot,t),w_t(\cdot,t),X(t))\|_{H_0^1(0,1)\times L^2(0,1)\times\mathbb{R}^n}\le\gamma e^{-\omega t}\|(w_0(\cdot),w_1(\cdot),x_0)\|_{H_0^1(0,1)\times L^2(0,1)\times\mathbb{R}^n}. \quad (49)$$

Proof: For equation w in (4), Smyshlyaev, Cerpa, and Krstic (2010) and Cox and Zuazua (1994) have proved

$$\|(w(\cdot,t),w_t(\cdot,t))\|_{H_0^1(0,1)\times L^2(0,1)}\le\delta e^{-\sigma t}\|(w_0(\cdot),w_1(\cdot))\|_{H_0^1(0,1)\times L^2(0,1)},\quad\exists\sigma,\delta>0,\quad\forall t>0.$$

In $X'(t)=(A+BK)X(t)+Bw_x(0,t)$, we cannot understand the term $w_x(0,t)$ in the sense of trace, because $w(\cdot,t)\in H^1(0,1)$, it is meaningless for $w_x(0,t)$. In fact, $w_x(0,t)$ is square integrable, which is known as 'hidden' regularity of wave equation (Lasiecka, Lions, and Triggiani 1986). Here, we can explain it by spectral approach. Because the functions $\{\sqrt{2}\sin(k\pi x)\}_{k\in\mathbb{N}}$ form a normal orthogonal basis of $L^2(0,1)$, $\{\frac{\sqrt{2}}{k\pi}\sin(k\pi x)\}_{k\in\mathbb{N}}$ form a normal orthogonal basis of $H_0^1(0,1)$, we can expand $w_0(x)$ and $w_1(x)$ with

$$w_0(x)=\sum_{k\in\mathbb{N}}w_0^k\frac{\sqrt{2}}{k\pi}\sin(k\pi x),\quad w_1(x)=\sum_{k\in\mathbb{N}}w_1^k\sqrt{2}\sin(k\pi x),$$

where expansion coefficients $\{w_0^k\}_{k\in\mathbb{N}},\{w_1^k\}_{k\in\mathbb{N}}$ satisfy

$$\|w_0\|_{H_0^1(0,1)}=\left(\sum_{k\in\mathbb{N}}|w_0^k|^2\right)^{\frac{1}{2}},\quad\|w_1\|_{L^2(0,1)}=\left(\sum_{k\in\mathbb{N}}|w_1^k|^2\right)^{\frac{1}{2}},$$

In system (4), we can obtain the solution

$$w(x,t)=\sum_{k\in\mathbb{N}}e^{-dt}\left\{w_0^k\cos(\alpha_k t)+\left(w_1^k+\frac{dw_0^k}{k\pi}\right)\sin(\alpha_k t)\right\}\times\frac{\sqrt{2}}{k\pi}\sin(k\pi x)$$

with $\alpha_k=\sqrt{k^2\pi^2+c}$. Then

$$w_x(0,t)=\sqrt{2}\sum_{k\in\mathbb{N}}e^{-dt}\left\{w_0^k\cos(\alpha_k t)+\left(w_1^k+\frac{dw_0^k}{k\pi}\right)\sin(\alpha_k t)\right\}, \quad (50)$$

obviously, $w_x(0,t)$ is exponential decay. According to (50), $A+BK$ be Hurwitz and the second equation of (4), we know $X(t)$ is exponential decay.

Therefore, there exists ω, $\gamma > 0$ such that the solution of (4) satisfies

$$\|(w(\cdot,t), w_t(\cdot,t), X(t))\|_{H_0^1(0,1)\times L^2(0,1)\times\mathbb{R}^n} \le \gamma e^{-\omega t}\|(w_0(\cdot), w_1(\cdot), x_0)\|_{H_0^1(0,1)\times L^2(0,1)\times\mathbb{R}^n},$$

which end the proof of Lemma 5.1.

Proof of Theorem 2.1: According to backstepping transformation (2) and system (4), setting $x=1$, we obtain

$$U(t) = \frac{1}{h(1)}\left\{\int_0^1 k(1,y)u(y,t)\mathrm{d}y + \int_0^1 s(1,y)u_t(y,t)\mathrm{d}y + M(1)X(t)\right\},$$

Under this controller, the solution of closed-loop system lies in the space $C([0,+\infty); H_L^1(0,1)) \cap C^1([0,+\infty); L^2(0,1)) \times C^1([0,+\infty))$ by standard arguments of operator semigroup theory. At the same time, under this controller, according to Lemma 5.1 and

$$\|(u(\cdot,t), u_t(\cdot,t), X(t))\|_{H^1(0,1)\times L^2(0,1)\times\mathbb{R}^n} \le \rho\|(w(\cdot,t), w_t(\cdot,t), X(t))\|_{H^1(0,1)\times L^2(0,1)\times\mathbb{R}^n}, \quad \exists\rho > 0,$$

in inverse transformation (41), we know the closed-loop system is exponentially stable in the sense of the norm

$$\left(\int_0^1 u^2(x,t)\mathrm{d}x + \int_0^1 u_x^2(x,t)\mathrm{d}x + \int_0^1 u_t^2(x,t)\mathrm{d}x + \|X(t)\|^2\right)^{\frac{1}{2}},$$

which finish the proof of Theorem 2.1.

6. Further discussion

In future work, there are several extensions of the results in this article to pursue. Firstly, one would like to consider the stabilization of coupled wave-ODE system with Neumann boundary condition and general space memory kernel, from the design procedure presented in this article, it is clear that an extension to coupled wave-ODE system with Neumann boundary condition and general space memory kernel may not pose any difficulties. Secondly, one would consider the stabilization of coupled Schrödinger-ODE system, Euler Bernoulli-beam-ODE system or other coupled PDEs system. Finally, eliminating variable $X(t)$ in (1), we obtain wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = \lambda(x)u_t(x,t) + \beta(x)u(x,t) \\ \qquad\qquad + C\left(x_0 e^{At} + \int_0^t e^{A(t-\tau)}Bu_x(0,\tau)\mathrm{d}\tau\right), \\ u(0,t) = 0, \quad u(1,t) = U(t), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \end{cases}$$

with a very especial time memory kernel $e^{A(t-\tau)}B$ and inhomogeneous term Cx_0e^{At}, this system can be exponentially stabilized by boundary control shown in Theorem 2.1, however, the more important and interesting problem is the stabilization of wave and heat equation with general time memory kernel and inhomogeneous term which have mentioned in Ivanov and Pandolfi (2009), Pandolfi (2009) and the references therein, as far as we know, the stabilization controller design method in this article does not carry over trivially from space memory system to time memory system which maybe involve some new ideas.

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