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Dissipation Analysis and H_∞ Control of Stochastic Nonlinear Systems Based on Hamiltonian Realization

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Abstract— **This paper proposes a novel dissipation analysis and a constructive H_∞ control method for stochastic nonlinear systems. First, we put forward a sufficient condition for the dissipation of stochastic nonlinear systems by completing their Hamiltonian realization. The internal structure and energy property of the system is explored as well. Then, we show that the dissipation property is preserved for parallel and feedback interconnected dissipative stochastic Hamiltonian systems. Moreover, based on the dissipation property of the subsystems, a feedback dissipation controller is proposed for series interacted dissipative stochastic Hamiltonian systems. Finally, we propose an H_∞ controller based on the stochastic dissipative Hamiltonian realization of uncertain stochastic nonlinear systems and show that the Hamiltonian function can be chosen to construct a solution of Hamiltonian-Jacobi inequality. Numerical simulation results illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

Physical systems subjected to random disturbances, such as measurement noises, environment changes, sudden failure etc, can be modelled as stochastic nonlinear systems [1]. Because random disturbances can drive a stable deterministic system unstable, many researchers have paid attention to the analysis and synthesis of stochastic systems. Based on the stochastic Lyapunov method and stochastic La Salle's invariant principle, Deng et al [2] and Mao [3] proposed some sufficient conditions for the stability and asymptotically stability in probability of stochastic nonlinear systems. Florchinger [4] extended the control Lyapunov function method of deterministic nonlinear systems to stochastic nonlinear systems and constructed a state feedback stabilization controller. Deng et al [5] applied the backstepping method to design stabilization controller of stochastic nonlinear systems driven by noise of unknown covariance. For the robust control of stochastic nonlinear systems, Niu et al [6] discussed the disturbance attenuation control of stochastic nonlinear systems and put forward a solvability condition for Hamiltonian-Jacobi inequality. In [8] and [9], nonlinear stochastic H_∞ controllers were constructed for stochastic nonlinear systems by solving the Hamiltonian-Jacobi equations or inequalities.

Generally, it is difficult to find a suitable Lyapunov function in stability analysis and controller design of stochastic nonlinear systems. Solving the Hamiltonian-Jacobi equations and inequality is also a difficult task. Observing that passivity and dissipation property has natural relationship with the stability and the storage function can always serve as a solution of Hamiltonian-Jacobi inequality, many researchers made much effort on the extension of the classical results to stochastic nonlinear systems. By using passive system approach, Florchinger [10] provided some sufficient conditions for the asymptotical stability of stochastic nonlinear systems. The passivity-based stabilization controllers were constructed as well. Lin et al [11] investigated the problem of stochastic passivity, feedback passivity and stabilization of stochastic nonlinear systems. In [12], Wu et al proposed some criteria on the existence and stability of time varying stochastic nonlinear systems by applying the dissipation theory. Ferreira et al [13] put forward some sufficient conditions for the stability in probability and noise-to-state stability of large-scale nonlinear stochastic systems by using the stochastic passivity properties of subsystems. However, the key question on how to construct a storage function to complete the dissipation or passivity based stability analysis and control still largely remains open.

Hamiltonian function method views the considered systems as composed of energy storage, dissipation and transformation components and utilizes their internal structure property to complete stability analysis and feedback controller design, see [14]- [16] and the references therein. One of the most important advantages of the Hamiltonian system method is that the Hamiltonian function can be chosen as a Lyapunov function candidate. Moreover, one can use the Hamiltonian function to construct a solution for Hamiltonian-Jacobi inequality. In this paper, we discuss the dissipation and H_∞ control of stochastic nonlinear system by transforming them to an equivalent Hamiltonian system, i.e., by completing their Hamiltonian realization. First, the passivity of stochastic nonlinear systems is discussed based on their Hamiltonian realization. The internal structure and energy property of the system is explored as well. Then, noticing that complex stochastic systems can be generally decomposed into smaller subsystems, we analyze the dissipation of parallel and feedback interconnected stochastic Hamiltonian systems and show that the dissipation property is preserved if the subsystems are dissipative. It is also shown that

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though the dissipation of series interconnected dissipative Hamiltonian systems is not preserved, a feedback dissipation controller can be constructed to drive the composed system dissipative by utilizing the dissipation characteristics of the subsystems. Finally, we discuss the L_2 gain and propose an H_∞ controller for stochastic nonlinear system with external disturbances. It is shown that the Hamiltonian function can be chosen to construct a solution of Hamiltonian-Jacobi inequality. A numerical example is given to demonstrate the effectiveness of the proposed method.

The remainder of this paper is organized as follows. In Section II, the dissipation of stochastic nonlinear systems is discussed based on their Hamiltonian realization. The dissipation and feedback dissipation of interconnected dissipative stochastic Hamiltonian systems are investigated in Section III. In section IV, we discuss the L_2 property of stochastic nonlinear systems and put forward an H_∞ controller for the system. Section V gives an example to verify the effectiveness of the proposed method. Finally, Section VI summarizes the paper and draws the conclusion.

II. Dissipation Analysis of Stochastic Nonlinear Systems Based on Hamiltonian Realization

Consider the following stochastic nonlinear systems described by the Itô stochastic differential equation

$$\begin{cases} dx = f(x)dt + g(x)udt + \tilde{g}(x)dw, \\ y = h(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u(t)$, $y(t) \in \mathbb{R}^m$ are, respectively, the state, the control input and the output. The signal $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space$(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. $f(x)$, $g(x)$ and $\tilde{g}(x)$ are matrix valued functions with proper dimensions. We suppose that $x = 0$ is an equilibrium point. We also suppose that the input u is a $\mathbb{R}^m$-valued measurable function and satisfies $E\left[\int_0^t \left\|u^2(s)\right\| ds\right] < \infty$ with respect to the measure $\mathcal{P}$, denoted by $E[\cdot]$.

Definition 2.1: Suppose there exists a continuous differentiable function $H(x)$ such that the system (1) can be reformulated as

$$\begin{cases} dx = \left[\big(J(x) - R(x)\big)\dfrac{\partial H(x)}{\partial x} + g(x)u\right] dt + \tilde{g}(x)dw, \\ y = g^T(x)\dfrac{\partial H(x)}{\partial x}, \end{cases} \tag{2}$$

where the structure matrix $J(x) \in \mathbb{R}^{(n\times n)}$ is skew-symmetric and the dissipation matrix $R(x) \in \mathbb{R}^{(n\times n)}$ is symmetric and positive semi-definite. Representation (2) is called a Hamiltonian realization of (1) and $H(x)$ is the corresponding Hamiltonian function.

Remark 2.1: The key to reformulating the system (1) as a stochastic Hamiltonian system is to find a continuous differentiable function $H(x)$ and structure matrix $R(x)$ and $J(x)$ such that $f(x) = \big(J(x) - R(x)\big)\dfrac{\partial H(x)}{\partial x}$. There are many methods to complete the Hamiltonian realization. One can refer to [14] for example.

For stochastic Hamiltonian system (2), we have the following theorem:

Theorem 2.1: Suppose the Hamiltonian function $H(x) > 0$ for all $x \neq 0$. The stochastic Hamiltonian system (2) is dissipative with respect to $H(x)$ if the following inequality holds:

$$-\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} + \frac{1}{2}\mathrm{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\} \leq 0. \tag{3}$$

What is more, the system is strictly dissipative if

$$-\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} + \frac{1}{2}\mathrm{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\} < 0. \tag{4}$$

holds.

Proof: Choose $H(x)$ as the storage function and the $y^T u$ as the supply rate. Direct calculation shows that the infinitesimal generator of the system

$$\begin{aligned} \mathcal{L}H(x) = &-\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} \\ &+\frac{1}{2}\mathrm{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\} + y^T u \leq y^T u, \end{aligned} \tag{5}$$

if (3) holds. So the system is dissipative. Moreover, if (4) holds, we have $\mathcal{L}H(x) < y^T u$ and the system is strict dissipative.

Remark 2.2: Theorem 2.1 is an extension of the (strict) dissipation of deterministic Hamiltonian systems. In fact, if the noise $w(t)$ does not exist, i.e., $\tilde{g}(x) = 0$, the stochastic Hamiltonian system reduces to the following deterministic Hamiltonian systems

$$\begin{cases} \dot{x} = \big(J(x) - R(x)\big)\dfrac{\partial H(x)}{\partial x} + g(x)u, \\ y = g^T(x)\dfrac{\partial H(x)}{\partial x}, \end{cases} \tag{6}$$

and (3) and (4) reduce to $-\dfrac{\partial^T H(x)}{\partial x}R(x)\dfrac{\partial H(x)}{\partial x} \leq 0$ and $-\dfrac{\partial^T H(x)}{\partial x}R(x)\dfrac{\partial H(x)}{\partial x} < 0$ respectively (or $R(x) \geq 0 (> 0)$ equivalently), which is exactly the sufficient conditions of dissipation and strict dissipation of deterministic Hamiltonian systems.

Generally, the Hamiltonian function $H(x)$ represents the total energy in the system. For deterministic Hamiltonian systems without control input, dissipation means that the total energy in the system is non-increasing and strict dissipation means strict decreasing. For stochastic Hamiltonian systems (2), the derivative of $H(x)$ along the trajectories can be written as

$$dH(x) = \mathcal{L}H(x) + \frac{\partial H(x)}{\partial x}\tilde{g}(x)dw, P - a.s.. \tag{7}$$

So we have

$$\begin{aligned} H(x(t)) = H(x(0)) &+ \int_0^t \mathcal{L}H(x(s))ds \\ &+ \int_0^t \frac{\partial H(x)}{\partial x}\tilde{g}(x)dw, P - a.s.. \end{aligned} \tag{8}$$

Taking expectation of the both sides and noticing that $E\left\{\int_0^t \frac{\partial H(x)}{\partial x}\tilde{g}(x)dw\right\} = 0$, we get

$$E\{H(x(t))\} = E\{H(x(0))\} + E\left\{\int_0^t y^T u ds\right\} - E\left\{\int_0^t \frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x}ds\right\} + E\left\{\int_0^t \frac{1}{2}\text{Tr}\left\{\tilde{g}(x)^T \frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\}ds\right\}. \quad (9)$$

While for the deterministic Hamiltonian system (6), the time derivative of the total energy $H(x)$ along the trajectories is

$$\dot{H}(x) = -\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} + y^T u, \quad (10)$$

and the integral form is

$$H(x(t)) = H(x(0)) - \int_0^t \frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x}ds + \int_0^t y^T u ds. \quad (11)$$

Comparing (11) with (9) one can find that the term $E\left\{\int_0^T \frac{1}{2}\text{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\}ds\right\}$ represents the expected variation of $H(x)$ caused by noise.

Actually, for the stochastic Hamiltonian system (2), the noise can be regarded as sources inside the system. To be more explicit, suppose that $\frac{1}{2}\text{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\}$ can be written as

$$\frac{1}{2}\text{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\} = \frac{\partial^T H(x)}{\partial x}S(x)\frac{\partial H(x)}{\partial x}, \quad (12)$$

where $S(x)$ is a proper matrix. Then we have

$$\mathcal{L}H = -\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} + y^T u + \frac{\partial^T H(x)}{\partial x}S(x)\frac{\partial H(x)}{\partial x}. \quad (13)$$

The above equation represents explicitly that the change of energy in the system can be divided into three parts:

(i) energy dissipation of the system, i.e., $-\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x}$, which is caused by the dissipative elements such as friction of mechanical systems and resistance of electrical systems;

(ii) energy injected through input and output ports, i.e., $y^T u$;

(iii) energy generated by the noise, i.e., $\frac{\partial^T H(x)}{\partial x}S(x)\frac{\partial H(x)}{\partial x}$, which is the energy change caused by the internal random noise in the system.

So the stochastic Hamiltonian system is a open system with internal energy dissipation, energy transformation, internal noise sources and energy injection from the input and output ports. The dissipation of the stochastic Hamiltonian system means that the energy stored in the system will not increase if there is no energy injection from outside.

Before the end of this subsection, we discuss the relationship between the dissipation and stability of stochastic Hamiltonian system. It is known that for deterministic Hamiltonian systems, if the Hamiltonian function achieves minimum at the equilibrium point, dissipation means stability and strict dissipation means asymptotically stability. The result also holds for stochastic Hamiltonian systems.

Theorem 2.2: Consider stochastic Hamiltonian system (2). Suppose it is dissipative and the Hamiltonian function $H(x)$ is non-negative with $H(x) \geq H(0) = 0$. Then the trivial solution of the system is stable in probability. Moreover, the strict dissipation implies asymptotically stable in probability.

Proof: Taking the Hamiltonian function $H(x)$ as a stochastic Lyapunov function and noticing that

$$\mathcal{L}(H(x)) = -\frac{\partial^T H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x} + \frac{1}{2}\text{Tr}\left\{\tilde{g}(x)^T\frac{\partial^2 H(x)}{\partial x}\tilde{g}(x)\right\}, \quad (14)$$

we can get the result directly.

III. Dissipation and Feedback Dissipation of Interconnected Stochastic Dissipative Hamiltonian Systems

Complex stochastic systems are always composed by several parallel, feedback or series interconnected subsystems. In this section, we discuss the dissipation and feedback dissipation of interconnected stochastic dissipative Hamiltonian systems, i.e., the following two (strict) dissipative stochastic Hamiltonian systems $\Sigma_i (i = 1, 2)$:

$$\Sigma_i : \begin{cases} dx_i = \left[\big(J_i(x_i) - R_i(x_i)\big)\frac{\partial H_i(x_i)}{\partial x_i}\right]dt \\ \qquad + g_i(x_i)u_1 dt + \tilde{g}_i(x_i)dw_i, \\ y_i = g_i^T(x_i)\frac{\partial H_i(x_i)}{\partial x_i}, \end{cases} \quad (15)$$

and investigate the dissipation of the composed system with different interconnected structures.

A. Dissipation of Parallel Interconnected Dissipative Stochastic Hamiltonian systems

Suppose Σ_1 and Σ_2 are parallel interconnected with $u_1 = u_2 = u$ and $y = y_1 + y_2$. We have the following result:

Theorem 3.1: The parallel interacted (strict) dissipative stochastic Hamiltonian system is (strict) dissipative.

Proof: The composed system can be written as

$$\begin{cases} dx_p = \begin{bmatrix} J_1 - R_1 & 0 \\ 0 & J_2 - R_2 \end{bmatrix}\frac{\partial H_p}{\partial x_p} \\ \qquad + g_p u + \begin{bmatrix} \tilde{g}_1 & 0 \\ 0 & \tilde{g}_2 \end{bmatrix}dw, \\ y = g_p^T\frac{\partial H_p}{\partial x_p}, \end{cases} \quad (16)$$

where $x_p = (x_1^T, x_2^T)^T$, $g_p(x) = (g_1^T(x_1), g_2^T(x_2))^T$, $dw = (dw_1^T, dw_2^T)^T$, $H_p(x) = H_1(x_1) + H_2(x_2)$. Then we have

$$\mathcal{L}H_p = \frac{\partial^T H_1}{\partial x_1}(J_1 - R_1)\frac{\partial H_1}{\partial x_1} + y_1^T u + \frac{1}{2}\text{Tr}\left\{\tilde{g}_1^T\frac{\partial^2 H_1}{\partial x_1^2}\tilde{g}_1\right\} + \frac{\partial^T H_2}{\partial x_2}(J_2 - R_2)\frac{\partial H_2}{\partial x_2} + y_2^T u + \frac{1}{2}\text{Tr}\left\{\tilde{g}_2^T\frac{\partial^2 H_2}{\partial x_2}\tilde{g}_2\right\}$$

From the (strict) dissipation of Σ_1 and Σ_2, we know that the parallel combination system is (strict) dissipative.

B. Dissipation of Feedback Interconnected Dissipative Stochastic Hamiltonian Systems

Now consider two feedback interconnected (strict) dissipative Hamiltonian systems Σ_1 and Σ_2 with $u_1 = y_2$ and $u_2 = -y_1$.

Theorem 3.2: The feedback interconnected (strict) dissipative stochastic Hamiltonian system is (strict) dissipative.

Proof: The composed system can be written as

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{bmatrix} J_1 - R_1 & g_1 g_2^T \\ -g_2 g_1^T & J_2 - R_2 \end{bmatrix} \begin{pmatrix} \dfrac{\partial H_1}{\partial x_1} \\ \dfrac{\partial H_2}{\partial x_2} \end{pmatrix} dt + \begin{bmatrix} \tilde{g}_1 & 0 \\ 0 & \tilde{g}_2 \end{bmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix}. \tag{17}$$

Let $H_f = H_1(x_1) + H_2(x_2)$, we have

$$\begin{aligned} \mathcal{L}H_f = &-\frac{\partial^T H_1}{\partial x_1} R_1 \frac{\partial H_1}{\partial x_1} - \frac{\partial^T H_2}{\partial x_2} R_2(x_2) \frac{\partial H_2}{\partial x_2} \\ &+\frac{1}{2}\mathrm{Tr}\left\{ \tilde{g}_1{}^T \frac{\partial^2 H_1}{\partial x_1^2} \tilde{g}_1 \right\} + \frac{1}{2}\mathrm{Tr}\left\{ \tilde{g}_2^T \frac{\partial^2 H_2}{\partial^2 x_2} \tilde{g}_2 \right\}. \end{aligned} \tag{18}$$

Because the systems Σ_1 and Σ_2 are (strictly) dissipative, we can see that their feedback interacted system is (strictly) dissipative.

C. Dissipation of Series Interconnected Dissipative Stochastic Hamiltonian Systems

We have seen that the passivity property preserves for parallel and feedback interconnected dissipative Hamiltonian systems. While for the serial interconnected case, the dissipation of the composed system can not be derived directly. But we can design a feedback controller to make it dissipative.

Theorem 3.3: Consider two serial interconnected (strict) dissipative stochastic Hamiltonian systems Σ_1 and Σ_2 where $u_2 = y_1$:

Under the feedback controller

$$u_1 = -g_2^T \frac{\partial H_2}{\partial x_2}, \tag{19}$$

the composed system is (strict) dissipative.

Proof: Denote $H_s = H_1(x_1) + H_2(x_2)$, $x_s = (x_1^T, x_2^T)^T$, $w = (w_1, w_2)$ and $g = (g_1, 0)^T$, the serial system can be reformulated as

$$\begin{cases} dx_s = \begin{bmatrix} J_1 - R_1 & 0 \\ g_2 g_1^T & J_2 - R_2 \end{bmatrix} \dfrac{\partial H_s}{\partial x_s} dt + g u_1 dt \\ \qquad + \begin{bmatrix} \tilde{g}_1 & 0 \\ 0 & \tilde{g}_2 \end{bmatrix} dw \\ y_2 = \begin{pmatrix} 0 & g_2^T \end{pmatrix} \dfrac{\partial H_s}{\partial x_s}. \end{cases} \tag{20}$$

Substitute (19) into (20) and along the trajectories of the closed loop system, we have

$$\begin{aligned} \mathcal{L}H_s = &-\frac{\partial^T H_1}{\partial x_1} R_1 \frac{\partial H_1}{\partial x_1} + \frac{1}{2}\mathrm{Tr}\{\tilde{g}_1^T \frac{\partial^2 H_1}{\partial x^2} \tilde{g}_1\} \\ &-\frac{\partial^T H_2}{\partial x_2} R_2 \frac{\partial H_2}{\partial x_2} + \frac{1}{2}\mathrm{Tr}\{\tilde{g}_2^T \frac{\partial^2 H_2}{\partial x^2} \tilde{g}_2\}. \end{aligned} \tag{21}$$

So under the feedback controller (19), the series interconnected system is (strict) dissipative if the subsystems are (strict) dissipative.

Remark 3.1: Based on the above results, we can decompose a complex stochastic system into smaller parallel, feedback or series interconnected sub-systems, reformulate them as stochastic Hamiltonian systems and analyze the dissipation of subsystems. The dissipation of the whole dynamic system can be obtained by the dissipation of the sub-systems.

IV. L_2-GAIN ANALYSIS AND H_∞ ROBUST CONTROL OF STOCHASTIC NONLINEAR SYSTEMS

In this section, we consider the robust control of stochastic nonlinear systems with external disturbances. First, we address the L_2 gain analysis of unforced stochastic nonlinear systems. Then we propose an constructive H_∞ controller design method for stochastic nonlinear systems by completing the Hamiltonian realization and utilizing the dissipation property.

A. L_2-gain Analysis of Stochastic Nonlinear Systems

Consider the following stochastic nonlinear system

$$\begin{cases} dx = f(x)dt + g_v(x)vdt + \tilde{g}(x)dw, \\ z = h(x), \end{cases} \tag{22}$$

where $v \in \mathbb{R}^r$ is the unknown bounded energy disturbance signal, $g_v \in \mathbb{R}^{n \times r}$ is a matrix valued function, $z \in \mathbb{R}^q$ is an estimation variable. As to other variables we refer to system (1).

Definition 4.1: For a given $\gamma > 0$, (22) is said to have L_2 gain no more than γ if

$$E\left(\int_0^T \|z\|^2 dt\right) \le E\left(\gamma^2 \int_0^T \|v\|^2 dt\right), \tag{23}$$

for all $v \in L_2[0, T]$.

Theorem 4.1: Consider the stochastic nonlinear system (22) with a given $\gamma > 0$. Let $V(x)$ be a non-negative definite differentiable function $V(x) \ge 0$ that satisfies $V(0) = 0$. If the following Hamiltonian-Jacobi inequality holds:

$$\begin{aligned} &\frac{\partial^T V}{\partial x} f + \frac{1}{2\gamma^2} \frac{\partial^T V}{\partial x} g_v g_v^T \frac{\partial V}{\partial x} \\ &+\frac{1}{2} z^T z + \frac{1}{2} \tilde{g}^T \frac{\partial^2 V}{\partial x^2} \tilde{g} \le 0, \end{aligned} \tag{24}$$

the L_2 gain of the system (22) is no more than γ.

Proof: Let $V(x)$ be a differentiable scalar function satisfying (24), then

$$\begin{aligned} &\frac{\partial^T V}{\partial x} f + \frac{\partial^T V}{\partial x} g_v v + \frac{1}{2} \tilde{g}^T \frac{\partial^2 V}{\partial x^2} \tilde{g} \\ \le &-\frac{1}{2} z^T z - \frac{1}{2\gamma^2} \frac{\partial^T V}{\partial x} g_v g_v^T \frac{\partial V}{\partial x} + \frac{\partial^T V}{\partial x} g_v v \\ &-\frac{1}{2}\gamma^2 v^T v + \frac{1}{2}\gamma^2 v^T v \\ = &\frac{1}{2}\{\gamma^2 \|v\|^2 - \|z\|^2\} - \frac{1}{2}\left\| \gamma v - \frac{1}{\gamma} g_v^T \frac{\partial^T V}{\partial x} \right\|^2. \end{aligned}$$

Noticing that $\left\|\gamma v - \frac{1}{\gamma} g_v^T \frac{\partial^T H}{\partial x}\right\|^2 \geq 0$, we get

$$\frac{\partial^T V}{\partial x} f + \frac{\partial^T V}{\partial x} g_v v + \frac{1}{2}\tilde{g}^T \frac{\partial^2 V}{\partial x^2}\tilde{g} \leq \frac{1}{2}\{\gamma^2 \|v\|^2 - \|z\|^2\}.$$

So along the trajectories of the system, the Itô's integral is

$$E\{V(T)\} - E\{V(0)\} \leq E\left\{\frac{1}{2}\int_0^T (\gamma^2 \|v\|^2 - \|z\|^2) dt\right\}.$$

Because $V(x) \geq 0$ and $V(0) = 0$, the above inequality is equivalent to

$$E\left(\int_0^T \|z\|^2 dt\right) \leq E\left(\gamma^2 \int_0^T \|v\|^2 dt\right).$$

B. H_∞ Control of Dissipative Stochastic Nonlinear Systems

Consider the H_∞ control of the following stochastic nonlinear system with external disturbances

$$\begin{cases} dx = f(x)dt + g_u(x)udt + g_v(x)vdt + \tilde{g}(x)dw, \\ z = h(x), \end{cases} \tag{25}$$

where $v \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$ are, respectively, the unknown bounded energy stochastic disturbance signal and the estimation variable. $g_v(x)$ and g_u are matrix valued functions with proper dimensions. We suppose that for the homogeneous system (i.e., $v = 0$) $x = 0$ is an equilibrium point.

Suppose that (25) can be stochastic dissipative Hamiltonian realized as

$$\begin{cases} dx = \left[\big(J(x) - R(x)\big)\frac{\partial H(x)}{\partial x} + g_u(x)u\right] dt \\ \qquad + g_v(x)vdt + \tilde{g}(x)dw, \\ z = g_u^T(x)\frac{\partial H(x)}{\partial x}. \end{cases} \tag{26}$$

The objective of the H_∞ controller design of the system (25), or equivalently (26), is to seek a state feedback control law $u(x) = \alpha(x)$ with $\alpha(0) = 0$ such that the L_2 gain of the closed loop system is no more than the given $\gamma > 0$ and the homogeneous closed loop system is asymptotically stable in probability.

Theorem 4.2: Consider the system (26), suppose the origin is a strict minimum of Hamiltonian function $H(x)$. Then for the given positive disturbance attenuation level γ, if the following inequality holds:

$$K g_u(x) g_u^T(x) + \frac{1}{2\gamma^2}\left[g_u(x)g_u^T(x) - g_v(x)g_v^T(x)\right] > 0, \tag{27}$$

an H_∞ controller can be constructed as

$$u = -\left[K + \frac{1}{2} + \frac{1}{2\gamma^2}\right] g_u^T(x)\frac{\partial H}{\partial x}, \tag{28}$$

where K is a positive constant.

Proof: The proof will be divided into two parts: (i) the L_2 gain of the closed loop system is no more than γ; (ii) the homogeneous closed loop system is asymptotically stable in probability.

For the first part, the closed loop system under the feedback controller (28) is

$$\begin{cases} dx = \left[\big(J - R - g_u\left[K + \frac{1}{2} + \frac{1}{2\gamma^2}\right] g_u^T\big)\frac{\partial H(x)}{\partial x}\right] dt \\ \qquad + g_v(x)vdt + \tilde{g}(x)dw, \\ z = g_u^T(x)\frac{\partial H(x)}{\partial x}. \end{cases} \tag{29}$$

Along the trajectories of the closed loop system, we have

$$\begin{aligned} &\frac{\partial^T H}{\partial x}\big(J - R - g_u\left[K + \frac{1}{2} + \frac{1}{2\gamma^2}\right] g_u^T\big)\frac{\partial H(x)}{\partial x} \\ &+ \frac{1}{2\gamma^2}\frac{\partial^T H}{\partial x} g_v g_v^T \frac{\partial H}{\partial x} + \frac{1}{2}z^T z + \frac{1}{2}\tilde{g}^T\frac{\partial^2 H}{\partial x^2}\tilde{g} \\ = &-\frac{\partial^T H}{\partial x} R \frac{\partial H(x)}{\partial x} + \frac{1}{2}\tilde{g}^T \frac{\partial^2 H}{\partial x^2}\tilde{g} + \frac{1}{2\gamma^2}\frac{\partial^T H}{\partial x} g_v g_v^T \frac{\partial H}{\partial x} \\ &-(K + \frac{1}{2\gamma^2})\frac{\partial^T H}{\partial x} g_u g_u^T \frac{\partial H(x)}{\partial x} \leq 0 \end{aligned}$$

According to Theorem 4.1, the L_2 gain of the closed loop system is no more than γ.

As to the second part, choose the Lyapunov function as $V(x) = H(x) - H(0)$. Along the trajectories of the closed loop system in absence of disturbances v, we get

$$\begin{aligned} \mathcal{L}H = &-\frac{\partial^T H}{\partial x} R(x)\frac{\partial H}{\partial x} + \frac{1}{2}\mathrm{Tr}\left\{\tilde{g}^T(x)\frac{\partial^2 H}{\partial x^2}\tilde{g}(x)\right\} \\ &-\frac{\partial^T H}{\partial x} g_u(x)\left[K + \frac{1}{2} + \frac{1}{2\gamma^2}\right] g_u^T(x)\frac{\partial H}{\partial x} < 0 \end{aligned}$$

So the closed loop system under $v = 0$ is asymptotically stable in probability.

V. Numerical Example

Consider the following stochastic nonlinear system

$$\begin{cases} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_1^3 + x_1 x_2^2 \\ -x_1 x_2^2 - x_2^3\end{bmatrix} dt + \begin{bmatrix}x_1 \\ x_2\end{bmatrix} udt + \begin{bmatrix}x_1 \\ x_2\end{bmatrix} vdt \\ \qquad + \begin{bmatrix}x_1^2 & 0 \\ 0 & x_2^2\end{bmatrix} dw \\ y = h(x) = g_u^T(x)\frac{\partial H}{\partial x} \end{cases}$$

with an external disturbance $v = x_1^2 + x_2^2$. The system can be Hamiltonian realized with $J = \begin{bmatrix}0 & x_1 x_2 \\ -x_1 x_2 & 0\end{bmatrix}$, $R = \begin{bmatrix}x_1^2 & 0 \\ 0 & x_2^2\end{bmatrix}$, $H(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Moreover, direct calculation shows that

$$\begin{aligned} &\frac{\partial^T H}{\partial x}\left[J(x) - R(x)\right]\frac{\partial H}{\partial x} + \frac{1}{2}\mathrm{Tr}\left\{\tilde{g}^T(x)\frac{\partial^2 H}{\partial x^2}\tilde{g}(x)\right\} \\ = &-\frac{1}{2}x_1^4 - \frac{1}{2}x_2^4 < 0. \end{aligned}$$

So the system is strict dissipative in absence of disturbance v.

Suppose the initial value $\begin{bmatrix}x_1(0) \\ x_2(0)\end{bmatrix} = \begin{bmatrix}1 \\ 1\end{bmatrix}$, for the given $\gamma = 1$, the response of the system with control input $u = 0$ is shown in Fig.1.

Now consider the controller $u = -2(x_1^2+x_2^2)$ with $K=1$, we can show that

$$Kg_u(x)g_u^T(x) + \frac{1}{2\gamma^2}\left[g_u(x)g_u^T(x) - g_v(x)g_v^T(x)\right] \geq 0$$

For the homogeneous closed loop system, we have $\mathcal{L}H = -2x_1^2x_2^2 \leq 0$. The response of the closed loop system is shown in Fig. 2. Obviously, the closed loop system is asymptotically stable.

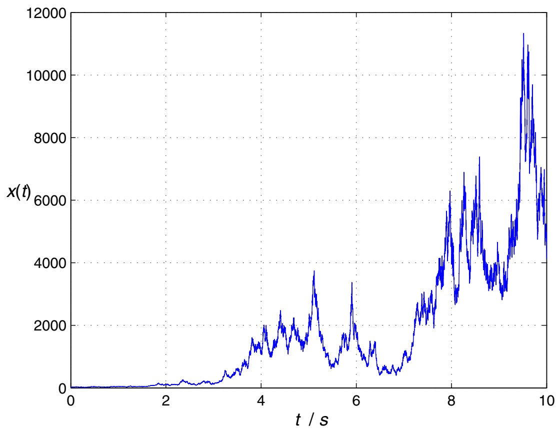

Fig. 1: the response of the system without control

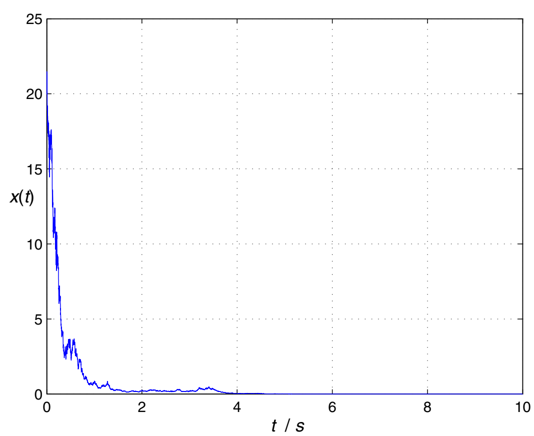

Fig. 2: the response of the system with H_∞ controller

VI. Conclusion

This paper investigates the dissipation and H_∞ control of stochastic nonlinear systems by the means of Hamiltonian system method. First, by completing the Hamiltonian realization of considered stochastic nonlinear systems, we propose a sufficient condition for the dissipation of the systems, where the Hamiltonian function is chosen as the storage function. The energy dissipation, transformation, internal source generation and energy exchange with external environment of the stochastic nonlinear systems is separated and become clear by reformulating them as stochastic Hamiltonian systems. Then, the dissipation and feedback dissipation of parallel, feedback and series interconnected dissipative stochastic Hamiltonian systems are discussed. We show that the dissipation property is preserved for parallel and feedback interconnected system. Though the dissipation of the series interconnected dissipative Hamiltonian systems is not reserved, we can construct a state feedback controller to drive the composed system dissipative by utilizing the internal structure of the subsystems. Finally, L_2 gain of stochastic nonlinear systems is analyzed and an H_∞ controller for stochastic nonlinear systems with external distrubances is constructed based on their Hamiltonian system structure and passivity property.

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