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Observer design for an IPDE with time-dependent coefficients

Shuxia Tang and Miroslav Krstic

Abstract— **This paper develops an exponentially convergent observer for a reaction-advection-diffusion integro-partial differential equation (IPDE) with time-dependent coefficients, via the PDE backstepping method. For the (I)PDEs with time-dependent coefficients, the backstepping transform gain kernel system is an (integral) evolution equation, and its coefficients also depend on time, which makes the derivation of the well-posedness of its solution to this (I)PDE nontrivial. The majorant argument is powerful in dealing with this difficulty, which is utilized in some existing literatures and is also employed in this study. To the best of the authors' knowledge, there are no existing results of observer design for the class of IPDEs with time-dependent, possibly unbounded, coefficients (which have possibly unbounded derivatives) on infinite time interval. Indeed, all the previous references for stabilization or observer design problem of an PDE with time-dependent coefficients consider a finite time interval, or the coefficients (and their derivatives) are required to be bounded with respect to the time variable. This paper could thus serve as a starting point for the study of these IPDEs.**

I. INTRODUCTION

Inspired by a state-of-charge (SoC) estimation problem for lithium-ion batteries, in which the PDE coefficients of the models are time-dependent [1], we consider a reaction-advection-diffuison IPDE with space-dependent and time-dependent coefficients. The problem is to design an observer for this system, and we employ the method of PDE backstepping.

PDE backstepping method is a systematic approach for stabilizing unstable (I)PDEs and designing observers for (I)PDEs, see [2], in which backstepping boundary controllers and observers are designed for some unstable parabolic, hyperbolic PDEs, etc.. Using the backstepping design, exponential stabilization of the resulting controlled system and exponential convergence of the resulting observer to the original system can be achieved. When applying this approach, the backstepping transformation kernel is required to satisfy another (I)PDE. For the (I)PDE system with time-dependent coefficients, this kernel (I)PDE is an (integral) evolution equation. Since its coefficients also depend on time, this introduces much difficulties into the discussion about the well-posedness of its solution.

There are a few existing literatures devoted to solving the stabilization or observer design problem of an PDE with time-dependent coefficients, and most of them deal with this difficulty by means of a majorant argument. The majorant technique is also utilized in this study, and the main difference and improvement from the previous results lies in the following two aspects. 1). Many previous references are considering a finite time interval. For example, [3], [4] and [5, Section 4] consider the problems related to reaction-(advection)-diffusion PDEs with a time-dependent reaction coefficient. But in this paper, we consider an infinite time interval, which is more general. In particular, the result can also be applied for the cases of finite time intervals. 2). More importantly, most of the previous results are based on some assumptions, such as the coefficients are inside some Gevrey class of functions, which are smooth but not necessarily analytic [6], [7]. In these cases, the coefficients (and their derivatives) are required to be bounded with respect to the time variable. For example, when the function $f(\cdot,t)$ is defined on an interval $(0,T)$, then for a fixed α, they require the existence of constants $Q,R>0$ such that for every positive integer k,

$$\sup_{t\in(0,T)}\left\|\frac{d^k f}{dt^k}\right\|\leq Q\frac{(k!)^\alpha}{R^k}.$$

This limitation is unlocked in this study, more precisely, we only require the above system coefficients to satisfy some majorant inequalities, for which the coefficients and their derivatives do not necessarily need to be bounded.

The outline of this paper is as follows. In Section *II*, the system under consideration is presented. In Section *III*, a backstepping state observer is designed and the observer error system is proved to be exponentially stable with an arbitrarily designated decay rate. It is worth noting that, because the kernel function system of the backstepping transform also has time-dependent coefficients, deriving existence and regularity of its solution is not trivial. Indeed, this is the main difficulty of solving our problem. Under some regularity assumptions and majorant arguments (dominant) of the system coefficients, existence and regularity of the solution to the system is proved in this section. Finally, some conclusion and possible future work are given in Section *IV*.

II. PROBLEM FORMULATION

We would like to consider a general one-dimensional linear reaction-advection-diffusion IPDE. Since the advection term can be transformed into zero [2, Section 4.8], we focus on considering the following equation with zero advection:

$$u_t(x,t)=u_{xx}(x,t)+c(x,t)u(x,t)+\int_0^x f(x,y,t)u(y,t)dy+g(x,t)u(0,t),\ x\in(0,1),\ t>0 \quad (1)$$

$$u_x(0,t)=j(t)u(0,t),\ t>0 \quad (2)$$

$$u_x(1,t)-e(t)u(1,t)=h(t)U(t),\ t>0 \quad (3)$$

$$u(x,0)=u_0(x),\ x\in[0,1], \quad (4)$$

S. Tang and M. Krstic are with Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093, USA. sht015@ucsd.edu; krstic@ucsd.edu.

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where $u(x,t)$, $x \in [0,1]$, $t \in [0,\infty)$ is the state variable; $U(t)$ is a (known) control input; $u(0,t)$ is the measured output. Our objective is to design an anti-collocated observer to reconstruct and online monitor the full state from the limited available information $u(0,t)$.

Remark 1: For the case when the system diffusivity is also dependent on both time and space evolution, we could employ a one-to-one analytic transformation to map the system into a system with a uniform coefficient 1, see, [3].

Assumption 1: All the system coefficient functions $c(x,t)$, $f(x,y,t)$, $g(x,t)$, $j(t)$, $e(t)$, $h(t)$ with $x \in [0,1], y \in [0,x], t \in [0,\infty)$ are known.

Assumption 2: The functions $c(x,t), f(x,y,t), e(t)$ in (1)-(4) are smooth and satisfy the following dominating inequalities:

$$|c(x,t)| \lll C_{c1}e^{C_{c2}t} \tag{5}$$

$$|f(x,y,t)| \lll C_{f1}e^{C_{f2}t} \tag{6}$$

$$|e(t)| \lll C_{e1}e^{C_{e2}t} \tag{7}$$

with respect to t, uniformly for x and y, in their respective sub-domains of the domain $\{(x,y,t) \mid 0 \le y \le x \le 1, t \ge 0\}$, where $C_{c1}, C_{c2}, C_{f1}, C_{f2}, C_{e1}, C_{e2}$ are positive constants.

Remark 2: For two functions $f_1(t), f_2(t)$ of t, the symbol $f_1 \lll f_2$ denotes the following relation between f_1 and f_2:

$$|f_1(t)| \le f_2(t); \quad \left|\frac{d^n f_1(t)}{dt^n}\right| \le \frac{d^n f_2(t)}{dt^n}, \ n = 1,2,\cdots. \tag{8}$$

Moreover, we call f_2 a dominant for f_1 (cf. [8]).

III. Observer design

A. Backstepping boundary observer design for the u-system

1) The $\hat{u}$-observer system and its convergence: Consider an anti-collocated Luenberger-type observer for the IPDE system $(1)-(4)$ with boundary state error injection:

$$\begin{aligned}\hat{u}_t(x,t) &= \hat{u}_{xx}(x,t) + c(x,t)\hat{u}(x,t) \\ &\quad + \int_0^x f(x,y,t)\hat{u}(y,t)dy + g(x,t)u(0,t) \\ &\quad + p_1(x,t)(u(0,t) - \hat{u}(0,t)), \ x \in (0,1), \ t > 0 \end{aligned} \tag{9}$$

$$\hat{u}_x(0,t) = j(t)u(0,t) + p_{10}(t)(u(0,t) - \hat{u}(0,t)), \ t > 0 \tag{10}$$

$$\hat{u}_x(1,t) - e(t)\hat{u}(1,t) = h(t)U(t), \ t > 0 \tag{11}$$

$$\hat{u}(x,0) = \hat{u}_0(x), \ x \in [0,1], \tag{12}$$

where the output injection functions $p_1(x,t)$ and $p_{10}(t)$ are to be determined. Then, the observer error

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t) \tag{13}$$

satisfies the following IPDE:

$$\begin{aligned}\tilde{u}_t(x,t) &= \tilde{u}_{xx}(x,t) + c(x,t)\tilde{u}(x,t) + \int_0^x f(x,y,t)\tilde{u}(y,t)dy \\ &\quad - p_1(x,t)\tilde{u}(0,t), \ x \in (0,1), \ t > 0 \end{aligned} \tag{14}$$

$$\tilde{u}_x(0,t) = -p_{10}(t)\tilde{u}(0,t), \ t > 0 \tag{15}$$

$$\tilde{u}_x(1,t) - e(t)\tilde{u}(1,t) = 0, \ t > 0 \tag{16}$$

$$\tilde{u}_0(x) = u_0(x) - \hat{u}_0(x), x \in [0,1]. \tag{17}$$

In order to find the suitable output injection gains, the PDE backstepping method [2] is employed. We would like to find an invertible continuous transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \int_0^x \tilde{p}(x,y,t)\tilde{u}(y,t)dy \tag{18}$$

so that the new variable $\tilde{w}$ satisfies the following exponentially stable system

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + \lambda\tilde{w}(x,t), \ x \in (0,1), \ t > 0 \tag{19}$$

$$\tilde{w}_x(0,t) = 0, \ t > 0 \tag{20}$$

$$\tilde{w}_x(1,t) + \frac{1}{2}\tilde{w}(1,t) = 0, \ t > 0, \tag{21}$$

where $\lambda < \frac{1}{4}$ is a free parameter which could be chosen to determine the observer's convergence rate.

Theorem 1: If choosing $\lambda < 1/4$, then for any initial value $\tilde{w}(\cdot,0) \in L^2(0,1)$, the $\tilde{w}$-system $(19)-(21)$ has an exponential stable (mild) solution $\tilde{w}(\cdot,t) \in L^2(0,1)$. If the boundary compatibility condition is also satisfied, then the $\tilde{w}$-system $(19)-(21)$ admits a classical solution.

Proof: Consider the Lyapunov function

$$E(t) = \frac{1}{2}\|\tilde{w}(\cdot,t)\|^2_{L^2(0,1)}, \tag{22}$$

then we could get

$$\dot{E}(t) \le -2\tilde{\rho}E(t), \ \tilde{\rho} = \frac{1}{4} - \lambda, \tag{23}$$

where the Poincaré inequality is employed. Thus, exponential stability of the $\tilde{w}$-system $(19)-(21)$ is proved with $\lambda < 1/4$. ■

By calculation and analysis, we get that the kernel function $\tilde{p}(x,y,t)$ in the transformation (18) needs to satisfy the following IPDE system:

$$\begin{aligned}\tilde{p}_t(x,y,t) &= \tilde{p}_{xx}(x,y,t) - \tilde{p}_{yy}(x,y,t) + (\lambda - c(y,t))\,\tilde{p}(x,y,t) \\ &\quad - \int_y^x \tilde{p}(x,\delta,t)f(\delta,y,t)d\delta + f(x,y,t) \end{aligned} \tag{24}$$

$$\tilde{p}(x,x,t) = \frac{\lambda}{2}(x-1) + \frac{1}{2}\int_x^1 c(y,t)dy + e(t) + \frac{1}{2} \tag{25}$$

$$\tilde{p}_x(1,y,t) = -\frac{1}{2}\tilde{p}(1,y,t), \tag{26}$$

for which the domain is $\mathscr{T} = \{(x,y,t) \mid 0 \le y \le x \le 1, t \ge 0\}$, and the output injection gains are chosen to satisfy

$$\begin{aligned} p_1(x,t) - \int_0^x \tilde{p}(x,y,t)p_1(y,t)dy &= -\tilde{p}_y(x,0,t) \\ &\quad - \tilde{p}(x,0,t)\left(\frac{\lambda}{2} - \frac{1}{2}\int_0^1 c(y,t)dy - e(t) - \frac{1}{2}\right) \end{aligned} \tag{27}$$

$$\begin{aligned} p_{10}(t) &= -\tilde{p}(0,0,t) \\ &= \frac{\lambda}{2} - \frac{1}{2}\int_0^1 c(y,t)dy - e(t) - \frac{1}{2}. \end{aligned} \tag{28}$$

2) Existence and regularity of the backstepping transformation and the observer output injection gains:

First, we prove the existence and regularity of the solution to the system (24)-(26), which gives the existence and regularity of the kernel function $\tilde{p}(x,y,t)$, and thus, also the existence and regularity of the transformation (18).

(Step One.) To derive the existence and regularity of the kernel function $\tilde{p}(x,y,t)$, we first transform the system $(24)-(26)$ into an equivalent integro-differential equation (IDE).

Let $\xi = x+y,\ \eta = x-y$ and $\tilde{q}(\xi,\eta,t) = \tilde{p}(x,y,t)$, then we have from $(24)-(26)$ that

$$\tilde{q}_t(\xi,\eta,t) = 4\tilde{q}_{\xi\eta}(\xi,\eta,t) + \left(\lambda - c\left(\frac{\xi-\eta}{2},t\right)\right)\tilde{q}(\xi,\eta,t) - \int_{(\xi-\eta)/2}^{(\xi+\eta)/2} \tilde{q}\left(\frac{\xi+\eta}{2}+\delta, \frac{\xi+\eta}{2}-\delta,t\right) f\left(\delta,\frac{\xi-\eta}{2},t\right)d\delta + f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2},t\right) \tag{29}$$

$$\tilde{q}(\xi,0,t) = \frac{\lambda}{2}\left(\frac{\xi}{2}-1\right) + \frac{1}{2}\int_{\xi/2}^{1} c(y,t)dy + e(t) + \frac{1}{2} \tag{30}$$

$$\tilde{q}_\xi(\xi,2-\xi,t) + \tilde{q}_\eta(\xi,2-\xi,t) = -\frac{1}{2}\tilde{q}(\xi,2-\xi,t), \tag{31}$$

that is,

$$\tilde{q}_t(\sigma,\tau,t) = 4\tilde{q}_{\sigma\tau}(\sigma,\tau,t) + \left(\lambda - c\left(\frac{\sigma-\tau}{2},t\right)\right)\tilde{q}(\sigma,\tau,t) - \int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2} \tilde{q}\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right) f\left(\delta,\frac{\sigma-\tau}{2},t\right)d\delta + f\left(\frac{\sigma+\tau}{2},\frac{\sigma-\tau}{2},t\right) \tag{32}$$

$$\tilde{q}(\sigma,0,t) = \frac{\lambda}{2}\left(\frac{\sigma}{2}-1\right) + \frac{1}{2}\int_{\sigma/2}^{1} c(y,t)dy + e(t) + \frac{1}{2} \tag{33}$$

$$\tilde{q}_\sigma(\sigma,2-\sigma,t) + \tilde{q}_\tau(\sigma,2-\sigma,t) = -\frac{1}{2}\tilde{q}(\sigma,2-\sigma,t). \tag{34}$$

From (32), we have

$$\tilde{q}_{\sigma\tau}(\sigma,\tau,t) = \frac{1}{4}\left[\tilde{q}_t(\sigma,\tau,t) - \left(\lambda - c\left(\frac{\sigma-\tau}{2},t\right)\right)\tilde{q}(\sigma,\tau,t) + \int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2} \tilde{q}\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right) f\left(\delta,\frac{\sigma-\tau}{2},t\right)d\delta - f\left(\frac{\sigma+\tau}{2},\frac{\sigma-\tau}{2},t\right)\right]. \tag{35}$$

Integrate (35) with respect to τ from 0 to η and use the boundary condition (33), then

$$\tilde{q}_\sigma(\sigma,\eta,t) = \frac{1}{4}\left(\lambda - c\left(\frac{\sigma}{2},t\right)\right) + \frac{1}{4}\int_0^\eta \left[\tilde{q}_t(\sigma,\tau,t) - \left(\lambda - c\left(\frac{\sigma-\tau}{2},t\right)\right)\tilde{q}(\sigma,\tau,t) + \int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2} \tilde{q}\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right) f\left(\delta,\frac{\sigma-\tau}{2},t\right)d\delta - f\left(\frac{\sigma+\tau}{2},\frac{\sigma-\tau}{2},t\right)\right]d\tau. \tag{36}$$

Integrate (36) with respect to σ from ξ to $2-\eta$, then we get

$$\tilde{q}(\xi,\eta,t) = \tilde{q}(2-\eta,\eta,t) - \frac{1}{4}\lambda(2-\xi-\eta) + \frac{1}{4}\int_\xi^{2-\eta} c\left(\frac{\sigma}{2},t\right)d\sigma - \frac{1}{4}\int_\xi^{2-\eta}\int_0^\eta \left[\tilde{q}_t(\sigma,\tau,t) - \left(\lambda - c\left(\frac{\sigma-\tau}{2},t\right)\right)\tilde{q}(\sigma,\tau,t) + \int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2} \tilde{q}\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right) f\left(\delta,\frac{\sigma-\tau}{2},t\right)d\delta - f\left(\frac{\sigma+\tau}{2},\frac{\sigma-\tau}{2},t\right)\right]d\tau d\sigma. \tag{37}$$

To find $\tilde{q}(2-\eta,\eta,t)$, from (31), we can derive

$$\frac{\partial}{\partial\xi}\tilde{q}(\xi,2-\xi,t) = 2\tilde{q}_\xi(\xi,2-\xi,t) + \frac{1}{2}\tilde{q}(\xi,2-\xi,t). \tag{38}$$

Let $\sigma = \xi, \eta = 2-\xi$ in (36), then (38) can be written in the form of an integro-differential equation for ξ as follows:

$$\frac{\partial}{\partial\xi}\tilde{q}(\xi,2-\xi,t) = \frac{1}{2}\tilde{q}(\xi,2-\xi,t) + \frac{1}{2}\left(\lambda - c\left(\frac{\xi}{2},t\right)\right) + \frac{1}{2}\int_0^{2-\xi}\left[\tilde{q}_t(\xi,\tau,t) - \left(\lambda - c\left(\frac{\xi-\tau}{2},t\right)\right)\tilde{q}(\xi,\tau,t) + \int_{(\xi-\tau)/2}^{(\xi+\tau)/2} \tilde{q}\left(\frac{\xi+\tau}{2}+\delta,\frac{\xi+\tau}{2}-\delta,t\right) f\left(\delta,\frac{\xi-\tau}{2},t\right)d\delta - f\left(\frac{\xi+\tau}{2},\frac{\xi-\tau}{2},t\right)\right]d\tau. \tag{39}$$

Let $\xi = 2$ in (30), then

$$\tilde{q}(2,0,t) = e(t) + \frac{1}{2}, \tag{40}$$

and thus, from (39), we have

$$\tilde{q}(\xi,2-\xi,t) = \left(e(t)+\frac{1}{2}\right)e^{\frac{1}{2}(\xi-2)} - \int_\xi^2 e^{\frac{1}{2}(\xi-s)}\left\{\frac{1}{2}\left(\lambda - c\left(\frac{s}{2},t\right)\right) + \frac{1}{2}\int_0^{2-s}\left[\tilde{q}_t(s,\tau,t) - \left(\lambda - c\left(\frac{s-\tau}{2},t\right)\right)\tilde{q}(s,\tau,t) + \int_{(s-\tau)/2}^{(s+\tau)/2} \tilde{q}\left(\frac{s+\tau}{2}+\delta,\frac{s+\tau}{2}-\delta,t\right) f\left(\delta,\frac{s-\tau}{2},t\right)d\delta - f\left(\frac{s+\tau}{2},\frac{s-\tau}{2},t\right)\right]d\tau\right\}ds, \tag{41}$$

and

$$\tilde{q}(2-\eta,\eta,t) = \left(e(t)+\frac{1}{2}\right)e^{-\frac{1}{2}\eta} - \int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)}\left\{\frac{1}{2}\left(\lambda - c\left(\frac{s}{2},t\right)\right) + \frac{1}{2}\int_0^{2-s}\left[\tilde{q}_t(s,\tau,t) - \left(\lambda - c\left(\frac{s-\tau}{2},t\right)\right)\tilde{q}(s,\tau,t) + \int_{(s-\tau)/2}^{(s+\tau)/2}\tilde{q}\left(\frac{s+\tau}{2}+\delta,\frac{s+\tau}{2}-\delta,t\right)f\left(\delta,\frac{s-\tau}{2},t\right)d\delta - f\left(\frac{s+\tau}{2},\frac{s-\tau}{2},t\right)\right]d\tau\right\}ds. \tag{42}$$

From (37) and (42), an IDE for $\tilde{q}(\xi,\eta,t)$ is obtained:

$$\tilde{q}(\xi,\eta,t) = \tilde{q}^0(\xi,\eta,t) + F[\tilde{q}](\xi,\eta,t), \tag{43}$$

where $\tilde{q}^0$ and $F[\tilde{q}]$ are defined by

$$\tilde{q}^0(\xi,\eta,t) = \left(e(t)+\frac{1}{2}\right)e^{-\frac{1}{2}\eta} - \frac{1}{2}\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)}\left[\lambda - c\left(\frac{s}{2},t\right) - \int_0^{2-s} f\left(\frac{s+\tau}{2},\frac{s-\tau}{2},t\right)d\tau\right]ds - \frac{1}{4}\lambda(2-\xi-\eta) + \frac{1}{4}\int_{\xi}^{2-\eta}\left[c\left(\frac{\sigma}{2},t\right) + \int_0^{\eta} f\left(\frac{\sigma+\tau}{2},\frac{\sigma-\tau}{2},t\right)d\tau\right]d\sigma \tag{44}$$

and

$$F[\tilde{q}](\xi,\eta,t) = -\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)} \times \frac{1}{2}\int_0^{2-s}\left[\tilde{q}_t(s,\tau,t) - \left(\lambda - c\left(\frac{s-\tau}{2},t\right)\right)\tilde{q}(s,\tau,t) + \int_{(s-\tau)/2}^{(s+\tau)/2}\tilde{q}\left(\frac{s+\tau}{2}+\delta,\frac{s+\tau}{2}-\delta,t\right) \times f\left(\delta,\frac{s-\tau}{2},t\right)d\delta\right]d\tau ds - \frac{1}{4}\int_{\xi}^{2-\eta}\int_0^{\eta}\left[\tilde{q}_t(\sigma,\tau,t) - \left(\lambda - c\left(\frac{\sigma-\tau}{2},t\right)\right)\tilde{q}(\sigma,\tau,t) + \int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2}\tilde{q}\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right) \times f\left(\delta,\frac{\sigma-\tau}{2},t\right)d\delta\right]d\tau d\sigma. \tag{45}$$

This IDE (43)-(45) is equivalent to the system (29)-(31).

(Step Two.) Next, we investigate the existence and regularity of the solution to (43)-(45). To begin with, set

$$\tilde{q}^{n+1}(\xi,\eta,t) = F[\tilde{q}^n](\xi,\eta,t), n = 0,1,2,\cdots, \tag{46}$$

then the series

$$\tilde{q}(\xi,\eta,t) = \sum_{n=0}^{\infty}\tilde{q}^n(\xi,\eta,t) \tag{47}$$

is a solution to the IDE (43)-(45). Thus, a proof of this series' convergence could show the existence of a solution $\tilde{q}(\xi,\eta,t)$.

Note that because of the partial derivative terms $\tilde{q}_t$ in the IDE (43)-(45), the properties of the series (47) could be quite complicated in general, and convergence may not hold in many cases. However, under the Assumption 2, the series could convergent to a continuous function in the domain, which is to be proved.

From Assumpiton 2, it can be proved that there exists a constant C such that

$$\left|q^0(\xi,\eta,t)\right| \ll Ce^{Ct} \tag{48}$$

with respect to t, uniformly for (ξ,η). In fact, we have

$$\begin{aligned}
&\left|\tilde{q}^0(\xi,\eta,t)\right| \\
&\le \left(C_{e1}e^{C_{e2}t}+\frac{1}{2}\right)e^{-\frac{1}{2}\eta} \\
&\quad + \frac{1}{2}\left[\left(|\lambda| + C_{c1}e^{C_{c2}t}\right)\eta + C_{f1}e^{C_{f2}t}\frac{\eta^2}{2}\right] \\
&\quad + \frac{1}{4}\left[\left(|\lambda| + C_{c1}e^{C_{c2}t}\right)(2-\xi-\eta) + C_{f1}e^{C_{f2}t}\eta(2-\xi-\eta)\right] \\
&\le C_{e1}e^{C_{e2}t} + \frac{1}{2} + |\lambda| + C_{c1}e^{C_{c2}t} + \frac{3}{4}C_{f1}e^{C_{f2}t} \\
&\le Ce^{Ct},
\end{aligned} \tag{49}$$

where

$$C = \max\left\{\frac{1}{2} + |\lambda| + C_{c1} + C_{e1} + C_{f1}, 1 + C_{c2} + C_{e2} + C_{f2}\right\}.$$

For any integer $m \ge 1$, we can also prove that

$$\begin{aligned}
&\left|\partial_t^m\tilde{q}^0(\xi,\eta,t)\right| \\
&\le C_{e1}C_{e2}^m e^{C_{e2}t}e^{-\frac{1}{2}\eta} \\
&\quad + \frac{1}{2}\left[C_{c1}C_{c2}^m e^{C_{c2}t}\eta + C_{f1}C_{f2}^m e^{C_{f2}t}\frac{\eta^2}{2}\right] \\
&\quad + \frac{1}{4}\left[C_{c1}C_{c2}^m e^{C_{c2}t}(2-\xi-\eta) + C_{f1}C_{f2}^m e^{C_{f2}t}\eta(2-\xi-\eta)\right] \\
&\le C_{e1}C_{e2}^m e^{C_{e2}t} + C_{c1}C_{c2}^m e^{C_{c2}t} + \frac{3}{4}C_{f1}C_{f2}^m e^{C_{f2}t} \\
&\le C^{m+1}e^{Ct} = \partial_t^m\left(Ce^{Ct}\right).
\end{aligned} \tag{50}$$

Assume that for any integer $n \ge 0$,

$$\left|\tilde{q}^n(\xi,\eta,t)\right| \ll C^{n+1}e^{(n+1)Ct}\left(\frac{3}{2}\right)^n\frac{(2-\xi)^n\eta^n}{n!}, \tag{51}$$

that is, for any integer $m \ge 0$,

$$\left|\partial_t^m\tilde{q}^n(\xi,\eta,t)\right| \le (n+1)^m C^{n+m+1}e^{(n+1)Ct}\left(\frac{3}{2}\right)^n\frac{(2-\xi)^n\eta^n}{n!}, \tag{52}$$

then, for any integer $m \geq 0$, from (46), we derive

$$
\begin{aligned}
&\left|\partial_t^m \tilde{q}^{n+1}(\xi,\eta,t)\right| \\
&\leq \frac{1}{2}\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)} \int_0^{2-s} \left|\partial_t^{m+1}\tilde{q}^n(s,\tau,t)\right| d\tau ds \\
&+\frac{1}{2}\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)} \int_0^{2-s} \left[|\lambda|\left|\partial_t^m \tilde{q}^n(s,\tau,t)\right| \right.\\
&\qquad \left. +\left|\partial_t^m\left(c\left(\frac{s-\tau}{2},t\right)\tilde{q}^n(s,\tau,t)\right)\right|\right] d\tau ds \\
&+\frac{1}{2}\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)} \int_0^{2-s}\int_{(s-\tau)/2}^{(s+\tau)/2} \\
&\quad \left|\partial_t^m\left[\tilde{q}^n\left(\frac{s+\tau}{2}+\delta,\frac{s+\tau}{2}-\delta,t\right) f\left(\delta,\frac{s-\tau}{2},t\right)\right]\right| d\delta d\tau ds \\
&+\frac{1}{4}\int_{\xi}^{2-\eta}\int_0^{\eta}\left|\partial_t^{m+1}\tilde{q}^n(\sigma,\tau,t)\right| d\tau d\sigma \\
&+\frac{1}{4}\int_{\xi}^{2-\eta}\int_0^{\eta}\left[|\lambda|\left|\partial_t^m\tilde{q}^n(\sigma,\tau,t)\right|\right. \\
&\qquad \left. +\left|\partial_t^m\left(c\left(\frac{\sigma-\tau}{2},t\right)\tilde{q}^n(\sigma,\tau,t)\right)\right|\right] d\tau d\sigma \\
&+\frac{1}{4}\int_{\xi}^{2-\eta}\int_0^{\eta}\int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2}\left|\partial_t^m\left[\tilde{q}^n\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right)\right.\right. \\
&\qquad\qquad \left.\left.\times f\left(\delta,\frac{\sigma-\tau}{2},t\right)\right]\right| d\delta d\tau d\sigma \\
&=\frac{1}{2}D_1+\frac{1}{4}D_2, \qquad (53)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &\triangleq \int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)}\int_0^{2-s}\left|\partial_t^{m+1}\tilde{q}^n(s,\tau,t)\right| d\tau ds \\
&+\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)}\int_0^{2-s}\left[|\lambda|\left|\partial_t^m\tilde{q}^n(s,\tau,t)\right|\right. \\
&\qquad \left.+\left|\partial_t^m\left(c\left(\frac{s-\tau}{2},t\right)\tilde{q}^n(s,\tau,t)\right)\right|\right] d\tau ds \\
&+\int_{2-\eta}^{2} e^{\frac{1}{2}(2-\eta-s)}\int_0^{2-s}\int_{(s-\tau)/2}^{(s+\tau)/2} \\
&\left|\partial_t^m\left[\tilde{q}^n\left(\frac{s+\tau}{2}+\delta,\frac{s+\tau}{2}-\delta,t\right) f\left(\delta,\frac{s-\tau}{2},t\right)\right]\right| d\delta d\tau ds,
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 &\triangleq \int_{\xi}^{2-\eta}\int_0^{\eta}\left|\partial_t^{m+1}\tilde{q}^n(\sigma,\tau,t)\right| d\tau d\sigma \\
&+\int_{\xi}^{2-\eta}\int_0^{\eta}\left[|\lambda|\left|\partial_t^m\tilde{q}^n(\sigma,\tau,t)\right|\right. \\
&\qquad \left.+\left|\partial_t^m\left(c\left(\frac{\sigma-\tau}{2},t\right)\tilde{q}^n(\sigma,\tau,t)\right)\right|\right] d\tau d\sigma \\
&+\int_{\xi}^{2-\eta}\int_0^{\eta}\int_{(\sigma-\tau)/2}^{(\sigma+\tau)/2}\left|\partial_t^m\left[\tilde{q}^n\left(\frac{\sigma+\tau}{2}+\delta,\frac{\sigma+\tau}{2}-\delta,t\right)\right.\right. \\
&\qquad\qquad \left.\left.\times f\left(\delta,\frac{\sigma-\tau}{2},t\right)\right]\right| d\delta d\tau d\sigma.
\end{aligned}
$$

From (52), we can derive the following bound:

$$
\begin{aligned}
D_1 &\leq (n+2)^m C^{n+m+2} e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{(n+1)!} \\
&\times \frac{2}{3Ce^{Ct}}\frac{1}{n+1}\left(\frac{n+1}{n+2}\right)^m\left[(n+1)C+|\lambda|+\sum_{i=0}^{m}\binom{m}{i}\right. \\
&\times\left.\left(C_{c1}e^{C_{c2}t}\left(\frac{C_{c2}}{(n+1)C}\right)^i+C_{f1}e^{C_{f2}t}\left(\frac{C_{f2}}{(n+1)C}\right)^i\frac{\eta}{n+2}\right)\right]. \qquad (54)
\end{aligned}
$$

Since it can be proved that

$$
\begin{aligned}
&(n+1)C+|\lambda|+\sum_{i=0}^{m}\binom{m}{i} \\
&\quad\times\left(C_{c1}e^{C_{c2}t}\left(\frac{C_{c2}}{(n+1)C}\right)^i+C_{f1}e^{C_{f2}t}\left(\frac{C_{f2}}{(n+1)C}\right)^i\frac{\eta}{n+2}\right) \\
&\leq 2Ce^{Ct}(n+1)\left(\frac{n+2}{n+1}\right)^m, \qquad (55)
\end{aligned}
$$

we get

$$
D_1 \leq \frac{4}{3}(n+2)^m C^{n+m+2}e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{(n+1)!}. \qquad (56)
$$

The following estimate can be obtained similarly:

$$
D_2 \leq \frac{4}{3}(n+2)^m C^{n+m+2}e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{(n+1)!}. \qquad (57)
$$

Therefore, from (53), (56) and (57), we have

$$
\begin{aligned}
&\left|\partial_t^m\tilde{q}^{n+1}(\xi,\eta,t)\right| \\
&\leq \frac{1}{2}D_1+\frac{1}{4}D_2 \\
&= (n+2)^m C^{n+m+2}e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{(n+1)!} \\
&= \partial_t^m\left(C^{n+2}e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{n+1!}\right), \qquad (58)
\end{aligned}
$$

which is equivalent to

$$
\left|\tilde{q}^{n+1}(\xi,\eta,t)\right| \lll C^{n+2}e^{(n+2)Ct}\left(\frac{3}{2}\right)^{n+1}\frac{(2-\xi)^{n+1}\eta^{n+1}}{n+1!}, \qquad (59)
$$

and thus, (51) is proved by induction.

From (51), we have that for an integer $n \geq 0$,

$$
\left|\tilde{q}^n(\xi,\eta,t)\right| \leq C^{n+1}e^{(n+1)Ct}\left(\frac{3}{2}\right)^n\frac{(2-\xi)^n\eta^n}{n!}, \qquad (60)
$$

and thus the series (47) is bounded by

$$
\tilde{q}(\xi,\eta,t) \leq Ce^{Ct}e^{\frac{3}{2}Ce^{Ct}(2-\xi)\eta}. \qquad (61)
$$

From the theorem of the comparison test, the series $\sum_{n=0}^{\infty}\tilde{q}^n(\xi,\eta,t)$ is absolutely and uniformly convergent with respect to (ξ,η). Hence, existence of $\tilde{q}(\xi,\eta,t)$ and also $\tilde{p}(x,y,t)$ is established. Moreover, $\tilde{p}(x,y,t)$ is $C^{\infty}(\mathscr{T})$ and

$$|\tilde{p}(x,y,t)| \leq Ce^{Ct}e^{\frac{3}{2}Ce^{Ct}(2-\xi)\eta}.$$

Then, the existence and regularity of the solution $p_1(x,t)$ to (27) can be obtained. Let us first write (27) into the following equivalent form:

$$p_1(x,t) = p_1^0(x,t) + G[p_1](x,t), \tag{62}$$

where

$$\begin{aligned} p_1^0(x,t) = &-\tilde{p}_y(x,0,t) \\ &-\tilde{p}(x,0,t)\left(\frac{\lambda}{2} - \frac{1}{2}\int_0^1 c(y,t)dy - e(t) - \frac{1}{2}\right), \end{aligned} \tag{63}$$

and

$$G[p_1](x,t) = \int_0^x \tilde{p}(x,y,t)p_1(y,t)dy. \tag{64}$$

Next, we could discuss about the solution to (62)-(64) in a similar way as the one for (43)-(45), which is omitted due to the space limitation.

3) Invertibility of the backstepping transformation: The transformation (18) could be proved to be invertible. Existence and regularity of its inverse could also be obtained. The proofs are omitted here.

4) Convergence of the designed observer: With the existence and regularity of kernel function for the transformation (18) and also from the invertibility of the transformation, the following main theorem of this paper could be proved.

Theorem 2: Under the Assumptions 1 and 2, if choosing $\lambda < 1/4$, then for any initial value $\tilde{u}_0(\cdot) \in L^2(0,1)$, the observer error $\tilde{u}$-system $(14)-(17)$, with the functions $p_1(x,t)$, $p_{10}(t)$ determined by (24)-(28), is exponentially stable. Therefore, for any initial value $u_0(\cdot)$, $\hat{u}_0(\cdot) \in L^2(0,1)$, the designed observer $(9)-(12)$, with the gain functions $p_1(x,t)$, $p_{10}(t)$ determined by (24)-(28), is exponentially convergent to the u-system $(1)-(4)$.

IV. Conclusion and Future Work

This paper discusses the problem of backstepping observer design for a reaction-advection-diffusion IPDE, and the designed observer is proved to be exponentially convergent.

The system coefficients are dependent on both spatial and time variables. We impose a more general assumption on the coefficients' regularity than the assumptions in the previous references. This regularity requirement unlocks the limitation on the boundedness of coefficients and their derivatives in the previous literatures, and thereby is more reasonable. To the best knowledge of the authors, this is the first systematic effort to deal with observer design for the class of IPDE systems with time-dependent, possibly unbounded, coefficients (which have possibly unbounded derivatives) on infinite time interval.

The PDE spatial domain in the consideration of this study is a fixed interval. Since the PDEs with time-dependent domain and PDEs with time-dependent coefficients are interchangeable [9], it is then also worth noting that this problem could be cast as a problem with time-dependent domain, i.e., moving boundary.

Our first next step is to consider the backstepping control design problem for this class of IPDEs. Moreover, we would like to emphasize that this work was inspired by a state-of-charge (SoC) estimation problem for lithium-ion batteries, which can be modeled to be a coupled PDE-ODE system with time-dependent coefficients [1]. Since the diffusivity in this model is also dependent on time and space, (I)PDEs with their coefficient of the highest spatial derivate [10] are our ongoing research subject as well. Another work direction orienting from this SoC estimation problem is to consider the observer design for some cascaded and coupled (I)PDE-ODE systems [11], [12], [13].

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