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STABILIZATION OF A HEAT-ODE SYSTEM CASCADED AT A BOUNDARY POINT AND AN INTERMEDIATE POINT

Zhiyuan Zhen, Shu-Xia Tang, and Zhongcheng Zhou

ABSTRACT

This paper considers the stabilization of a heat-ODE system cascaded at a boundary point and an intermediate point. The stabilizing feedback control law is designed by the backstepping method. Based on a novel transformation, we prove that all the kernel functions in the forward and inverse transformations are of the class C^2. Moreover, the effectiveness of controller design is shown with a numerical simulation. Finally, we show the coherence between the controllability assumption of the main theorem in this paper and the known one for a special case with $\lambda = 0$.

Key Words: Cascaded heat-ODE, backstepping, stabilization, intermediate point..

I. INTRODUCTION

This paper considers the exponential stabilization problem of a cascaded system with a heat partial differential equation (PDE) and an ordinary differential equation (ODE):

$$\begin{cases} \dot{X}(t) = AX(t) + B_0 u(0,t) + Bu(x_0,t), \\ u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u_x(0,t) = 0,\ u(1,t) = U(t), \\ u(x,0) = u_0(x), \\ X(0) = X_0, \end{cases} \tag{1}$$

where $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))^T \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times 1}$, $B \in \mathbb{R}^{n\times 1}$, $\lambda \in \mathbb{R}$, $x_0 \in (0,1)$, $(X(t),\ u(x,t))$ is the state and $U(t)$ is the boundary control.

Stabilization of PDEs is a fundamental problem in control theory, and there are many approaches to handle it, for example, control Lyapunov function, return method [1], the linear quadratic method [2] and backstepping method [3]. In this paper, we employ the backstepping method. In fact, the backstepping method, because of its simplicity and efficiency for feedback controller design, has become more and more popular in engineering applications. Many references can be found regarding the utilization of the backstepping method for stabilizing controller designs. For example, researchers have considered the stabilizing feedback controller designs of heat equations in [3–6], the cascaded PDE-ODE systems in [7–10] and the coupled PDE-ODE systems in [11–13].

In the system (1), a state boundary point value $u(0,t)$ and a state intermediate point value $u(x_0,t)$ of the heat equation enter the ODE system simultaneously. This system can be considered as the combined system of the models in [9] and [10]. If $X(t) = (X_1(t), X_2(t))$, this model can be considered as the output of heat equation at $x = 0$ and x_0 transferred into force entering the ODE system. At the same time, from the viewpoint of computation, the computation of $\dot{X}(t) = AX(t) + \int_0^1 B(y)u(y,t)dy$ in [7] needs to be approximated by $\dot{X}(t) = AX(t) + B_1 u(x_1,t) + B_2 u(x_2,t) + \cdots + B_n u(x_n,t)$ cascading with multiple points. The system (1) cascading with two points is exactly a special case. For the system (1), the special case of $B = 0$ has been considered in [9]. If $B_0 = 0$, $\lambda = 0$ and a special A, the problem has been considered in [10]. However, when $B \neq 0$ and $B_0 \neq 0$, the controller design procedure in [9,10] does not work any more and this will lead to some difficulties in designing the stabilizing feedback controller because $u(0,t)$ and $u(x_0,t)$ will impose on $X(t)$ equation simultaneously.

Note that if we set $B_1(x) = B\delta(x - x_0) + B_0\delta(x)$, $\delta(x)$ is the Dirac function, then the nonlocal source $\int_0^1 B_1(x)u(x,t)dx = B_0 u(0,t) + Bu(x_0,t)$ and then the system (1) become a system similar to the one in [7] with counter-convection term $-bu_x(x,t)$ replaced by reaction term $\lambda u(x,t)$. The first novelty of this paper lies in the general reaction term $\lambda u(x,t)$. It is well-known that the counter-convection term $-bu_x(x,t)$ in [7] does not introduce instability into the PDE system; and it

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Zhiyuan Zhen (e-mail: zhenzhiyuan@swu.edu.cn) and Zhongcheng Zhou (corresponding author, e-mail: zhouzc@amss.ac.cn) are with the School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China.
Shu-Xia Tang (e-mail: shuxiatang.control@gmail.com) is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA.
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can be easily deleted through a function transformation $v(x) = e^{-bx}u(x)$, which does not change the stability of original system. However, more effort is always needed to deal with the instability caused by the reaction terms. The second novelty lies in the transformation proposed in this paper. We add one more integral term in the backstepping transformation with undetermined kernels function instead of the partially pre-determined backstepping transformation, which can simplify the computation. Indeed, it is needed in [7] to solve a nonhomogeneous equation $g(x)$ with weak regularity term. However, in this paper, thanks to the modified transformation, we only need to solve two homogeneous equations for this part, and we are able to obtain C^2 kernel functions in both the direct transformation and its inverse instead of the continuous kernels function in [7], the C^2- smooth kernel function can be used to simplify the computation of controllability assumption. Meanwhile, for the system (1) with $\lambda = 0$, we obtain the controllability assumption in this paper and [7] is the same via different stabilization control design procedure. Moreover, for general λ in (1), we also obtain the stabilization feedback controller with C^2- smooth kernel function. This paper is organized as follows. In Section II, we discuss the stabilizing controller design. In Section III, we prove existence of smooth kernel functions in forward transformation (2), and the feedback boundary controller is obtained to stabilize the original system. In Section IV, we prove the existence of inverse transformation. Then, according to stability of the target system and the regularity of both the direct transformation and its inverse transformation, exponential stability of the closed-loop system is proved in Section V. In Section VI, we give the simulation of the closed-loop system. In Section VII, we give the conclusion and further works. Finally, in the appendix, we compare the controllability assumptions in this paper and in [7], and derive the coherence between the two.

II. CONTROL DESIGN

For obtaining a feedback stabilizing control law of the system (1), we adopt the following novel transformation $(u, X) \mapsto (w, Z)$ as

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u dy - \gamma(x)Z(t), \tag{2}$$

$$Z(t) = X(t) + \int_0^1 G(y)u dy + \int_0^{x_0} F(x)u dx \tag{3}$$

with undetermined kernels $k(x,y) \in \mathbb{R}$, $\gamma(x) \in \mathbb{R}^{1\times n}$, $G(y) \in \mathbb{R}^{n\times 1}$ and $F(x) \in \mathbb{R}^{n\times 1}$, to convert (1) into an exponentially stable target system

$$\begin{cases} \dot{Z}(t) = A_1 Z(t), \\ w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = 0, w(1,t) = 0, \\ w(x,0) = w_0(x), \\ Z(0) = Z_0, \end{cases} \tag{4}$$

where $A_1 = A - G'(1)K$ satisfies assumption **(H)**.
(H). Assume the system $(A, G'(1))$ is controllable, then there exists a K such that the matrix $A_1 = A - G'(1)K$ is Hurwitz.

From the transformation (2) and the boundary conditions in (1) and (4), we can obtain the stabilizing feedback control law

$$\begin{aligned} U(t) &= \gamma(1)Z(t) + \int_0^1 k(1,y)u(y,t)dy \\ &= KZ(t) + \int_0^1 k(1,y)u(y,t)dy, \end{aligned} \tag{5}$$

where we have set $\gamma(1) = K$. To prove the stability of the closed-loop system (1) and (5), we need to obtain the inverse transformation of (2), which is discussed in Section IV. Finally, we obtain the stability of closed-loop system (1) and (5).

III. SOLUTIONS TO THE KERNELS IN FORWARD TRANSFORMATION

Taking the derivative of (3) with respect to t, we obtain

$$\begin{aligned} &\dot{Z}(t) \\ &= \dot{X}(t) + \int_0^1 G(y)u_t(y,t)dy + \int_0^{x_0} F(x)u_t(x,t)dx \\ &= AX(t) + B_0 u(0,t) + Bu(x_0,t) + \int_0^1 G(y)u_{yy}dy \\ &\quad + \int_0^{x_0} F(x)u_{xx}dx + \int_0^1 \lambda G(y)u dy + \int_0^{x_0} \lambda F(x)u dx \\ &= (A - G'(1)K)Z(t) + \big(B_0 + G'(0) + F'(0)\big)u(0,t) \\ &\quad - \int_0^1 \big((A - \lambda I_n)G(y) + G'(1)k(1,y) - G''(y)\big)u dy \\ &\quad - \int_0^{x_0} ((A - \lambda I_n)F(x) - F''(x))u dx + F(x_0)u_x(x_0,t) \\ &\quad - (F'(x_0) - B)u(x_0,t) + G(1)u_x(1,t) \\ &= (A - G'(1)K)Z(t), \end{aligned}$$

by (1), (4), (5) choosing the kernel functions $G(\cdot)$, $F(\cdot)$ to satisfy

$$\begin{cases} G''(y)-(A-\lambda I_n)G(y)-G'(1)k(1,y)=0,\\ G(1)=0,\ G'(0)=-F'(0)-B_0,\\ F''(x)-(A-\lambda I_n)F(x)=0,\\ F(x_0)=0, F'(x_0)=B. \end{cases} \tag{6}$$

Then, taking the derivative of (2) with respect to x twice and t once, we obtain

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t)-k(x,x)u_x(x,t)\\ &\quad -\left(\frac{d}{dx}k(x,x)+k_x(x,x)\right)u(x,t)\\ &\quad -\int_0^x k_{xx}(x,y)udy-\gamma''(x)Z(t) \end{aligned} \tag{7}$$

and

$$\begin{aligned} & w_t(x,t)\\ =& u_t(x,t)-\int_0^x k(x,y)u_t(y,t)dy-\gamma(x)\dot{Z}(t)\\ =& u_{xx}(x,t)+\lambda u(x,t)-k(x,x)u_x(x,t)+k(x,0)u_x(0,t)\\ &+k_y(x,x)u(x,t)-k_y(x,0)u(0,t)-\int_0^x k_{yy}(x,y)udy\\ &-\int_0^x \lambda k(x,y)udy-\gamma(x)A_1Z(t), \end{aligned} \tag{8}$$

where the notations

$$k_x(x,x)=\frac{\partial k(x,y)}{\partial x}\Big|_{y=x},\ k_y(x,x)=\frac{\partial k(x,y)}{\partial x}\Big|_{y=x}$$

and

$$\frac{d}{dx}k(x,x)=k_x(x,x)+k_y(x,x)$$

are applied. Combining (7) with (8), we have

$$\begin{aligned} & w_t(x,t)-w_{xx}(x,t)\\ &=(2k'(x,x)+\lambda)u(x,t)+\big(\gamma''(x)-\gamma(x)A_1\big)Z(t)\\ &\quad+\int_0^x\big(k_{xx}(x,y)-k_{yy}(x,y)-\lambda k(x,y)\big)u(y,t)dy\\ &\quad-k_y(x,0)u(0,t), \end{aligned}$$

According to $w_t(x,t)-w_{xx}(x,t)=0$, we choose $k(x,y)$ and $\gamma(x)$ to satisfy

$$\begin{cases} k_{xx}(x,y)-k_{yy}(x,y)=\lambda k(x,y),\\ k'(x,x)=-\frac{\lambda}{2},\ \ k_y(x,0)=0,\\ \gamma''(x)-\gamma(x)A_1=0. \end{cases}$$

From $w_x(0,t)=0$, we have

$$w_x(0,t)=u_x(0,t)-k(0,0)u(0,t)-\gamma'(0)Z(t)=0.$$

Hence, we choose $\gamma'(0)=0$ and $k(0,0)=0$. Therefore, $G(y)$, $F(x)$, $k(x,y)$ and $\gamma(x)$ satisfy the following coupled equations

$$\begin{cases} G''(y)-(A-\lambda I_n)G(y)-G'(1)k(1,y)=0,\\ G(1)=0,\ \ G'(0)=-F'(0)-B_0,\\ F''(x)-(A-\lambda I_n)F(x)=0,\\ F(x_0)=0,\ \ F'(x_0)=B,\\ k_{xx}(x,y)-k_{yy}(x,y)=\lambda k(x,y),\\ k(x,x)=-\frac{\lambda}{2}x,\ \ k_y(x,0)=0,\\ \gamma''(x)-\gamma(x)A_1=0,\\ \gamma'(0)=0,\ \ \gamma(1)=K. \end{cases} \tag{9}$$

We first separate (9) into four subsystems of equations

$$\begin{cases} k_{xx}(x,y)-k_{yy}(x,y)=\lambda k(x,y),\\ k(x,x)=-\frac{\lambda}{2}x,\quad k_y(x,0)=0, \end{cases} \tag{10}$$

$$\begin{cases} G''(y)-(A-\lambda I_n)G(y)-G'(1)k(1,y)=0,\\ G(1)=0,\quad G'(0)=-F'(0)-B_0, \end{cases} \tag{11}$$

$$\begin{cases} F''(x)-(A-\lambda I_n)F(x)=0,\\ F(x_0)=0,\quad F'(x_0)=B \end{cases} \tag{12}$$

and

$$\begin{cases} \gamma''(x)-\gamma(x)A_1=0,\\ \gamma'(0)=0,\quad \gamma(1)=K. \end{cases} \tag{13}$$

then, we will solve $k(x,y)$, $G(y)$, $F(x)$ and $\gamma(x)$ separately. The equation (10) has a solution

$$k(x,y)=-\lambda x\frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}, \tag{14}$$

where I_1 is the first-order modified Bessel function of the first kind. Let

$$\mathbb{F}(x)=\begin{pmatrix} F(x)\\ F'(x)\end{pmatrix},\quad D=\begin{pmatrix} 0 & I_n\\ A-\lambda I_n & 0\end{pmatrix},$$

where I_n is the identity matrix, then (12) can be written as

$$\mathbb{F}'(x)=D\mathbb{F}(x),\ \ \mathbb{F}(x_0)=\begin{pmatrix}0\\ B\end{pmatrix}, \tag{15}$$

and thus

$$\mathbb{F}(x)=e^{D(x-x_0)}\mathbb{F}(x_0). \tag{16}$$

Therefore,

$$F(x)=\begin{pmatrix} I_n & 0\end{pmatrix}e^{D(x-x_0)}\begin{pmatrix}0\\ B\end{pmatrix}. \tag{17}$$

To solve (11), similarly, we obtain

$$G(x) = \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(x-1)} \begin{pmatrix} 0 \\ G'(1) \end{pmatrix} + \begin{pmatrix} I_n & 0 \end{pmatrix} \int_1^x e^{D(x-s)} \begin{pmatrix} 0 \\ k(1,s)G'(1) \end{pmatrix} ds, \tag{18}$$

where

$$M = \begin{pmatrix} k(1,0)I_n & (1 - \int_0^1 k(1,s)ds)I_n \end{pmatrix} e^{-D} \begin{pmatrix} 0 \\ I_n \end{pmatrix}$$
$$G'(1) = -M^{-1}(F'(0) + B_0), \tag{19}$$

and

$$F'(0) = \begin{pmatrix} 0 & I_n \end{pmatrix} e^{-Dx_0} \begin{pmatrix} 0 \\ B \end{pmatrix}. \tag{20}$$

The solution of (13) can also be solved as

$$\gamma(x) = K\Lambda^{-1} \begin{pmatrix} I_n & 0 \end{pmatrix} e^{\begin{pmatrix} 0 & A_1 \\ I_n & 0 \end{pmatrix} x} \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \tag{21}$$

where

$$\Lambda = \begin{pmatrix} I_n & 0 \end{pmatrix} e^{\begin{pmatrix} 0 & A_1 \\ I_n & 0 \end{pmatrix}} \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \tag{22}$$

Based on the above calculation and analysis, setting $\mathbb{T} := \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1,\ \ 0 \le y \le x\}$, we obtain the existence of the solutions to (9) as stated in following theorem.

Theorem 1. Assume the matrices M in (19) and Λ in (22) are invertible respectively, there exist classical solutions $k(\cdot,\cdot) \in C^2(\mathbb{T})$, $G(\cdot) \in C^2([0,1])$, $\gamma(\cdot) \in C^2([0,1])$ and $F(\cdot) \in C^2([0,x_0])$ to (9).

Remark 1. The invertibility of Λ is equivalent to an assumption that there are no eigenvalues of matrix A_1 located at the positions $-\frac{(2k+1)^2\pi^2}{4} (k \in \mathbb{Z}^+)$. The invertibility of M is dependent on A and λ in (1), but not dependent on the feedback matrix K. In fact, there are some A and λ that satisfy the assumption of Theorem 1. For example, for $n = 1$, we can check that M in (19) and Λ in (22) are invertible for some given value. In (4), if we choose $A_1 < 0$ and $A_1 \neq -\frac{(2k+1)^2\pi^2}{4} (k \in \mathbb{Z}^+)$, then

$$\Lambda = \begin{pmatrix} 1 & 0 \end{pmatrix} e^{\begin{pmatrix} 0 & A_1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\sqrt{-A_1}) \neq 0,$$

which is invertible. Similarly, in (1), if $A > 0$, then

$$M = \begin{pmatrix} 0 & 1 \end{pmatrix} e^{-\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}(e^{\sqrt{A}} + e^{-\sqrt{A}}),$$

which is also invertible.

IV. INVERSE TRANSFORMATION

In this section, we will show that the transformations (2) and (3) are invertible. First, we consider the invertibility of (2), which can be written as:

$$u(x,t) = w(x,t) + \int_0^x l(x,y)wdy + \varphi(x)Z(t), \tag{23}$$

where the functions $l(x,y)$ and $\varphi(x)$ are to be determined later. We will show the inverse transformation is well-defined after determining the kernel functions. Taking the derivative of u in (23) with respect to x and t, we have

$$\begin{aligned} & u_t(x,t) - u_{xx}(x,t) - \lambda u(x,t) \\ &= -(2l'(x,x) + \lambda)w(x,t) - \left(\varphi''(x) - \varphi(x)A_1\right)Z(t) \\ &\quad - \int_0^x \left(l_{xx}(x,y) - l_{yy}(x,y) + \lambda l(x,y)\right)w(y,t)dy \\ &\quad + l_y(x,0)w(0,t). \end{aligned} \tag{24}$$

According to $u_x(0,t) = 0$, it holds that

$$w_x(0,t) + l(0,0)w(0,t) + \varphi'(0)Z(t) = 0.$$

Hence, we get

$$l(0,0) = 0, \quad \varphi'(0) = 0. \tag{25}$$

Therefore, combining (1) and (24), we choose l and φ satisfy

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y), \\ l(x,x) = -\frac{\lambda}{2}x, \quad l_y(x,0) = 0 \end{cases} \tag{26}$$

and

$$\begin{cases} \varphi''(x) - \varphi(x)A_1 = 0, \\ \varphi'(0) = 0. \end{cases} \tag{27}$$

According to (10) and (26), we obtain that l and k satisfy

$$l(x,y) = k(x,y) + \int_y^x l(x,z)k(z,y)dz. \tag{28}$$

In terms of (23), we obtain

$$\begin{aligned} &u(1,t) \\ &= w(1,t) + \int_0^1 l(1,y)wdy + \varphi(1)Z(t) \\ &= \left(\varphi(1) - \int_0^1 l(1,y)\gamma(y)dy\right)Z(t) + \int_0^1 l(1,y)udy \\ &\quad - \int_0^1\int_0^y l(1,y)k(y,z)u(z,t)dzdy \\ &= \left(\varphi(1) - \int_0^1 l(1,y)\gamma(y)dy\right)Z(t) \\ &\quad + \int_0^1 \left(l(1,z) - \int_z^1 l(1,y)k(y,z)dy\right)u(z,t)dz \\ &= KZ(t) + \int_0^1 k(1,y)u(y,t)dy \end{aligned} \tag{29}$$

after taking

$$\varphi(1) = K + \int_0^1 l(1,y)\gamma(y)dy. \tag{30}$$

Therefore, we obtain

$$\begin{cases} \varphi''(x) - \varphi(x)A_1 = 0, \\ \varphi'(0) = 0, \ \ \varphi(1) = K + \int_0^1 l(1,y)\gamma(y)dy, \end{cases} \tag{31}$$

which can be solved explicitly if Λ is invertible. The explicit solution of (31) is

$$\begin{aligned} \varphi(x) = &\left(K + \int_0^1 l(1,y)\gamma(y)dy\right)\Lambda^{-1} \\ &\times \begin{pmatrix} I_n & 0 \end{pmatrix} e^{\begin{pmatrix} 0 & A_1 \\ I_n & 0 \end{pmatrix} x} \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \end{aligned} \tag{32}$$

Meanwhile, in terms of (2), we obtain

$$\begin{aligned} X(t) = &\left(1 - \int_0^1 G(x)\varphi(x)dx - \int_0^{x_0} F(x)\varphi(x)\mathrm{d}x\right)Z(t) \\ &- \int_0^1 G(x)w(x,t)\mathrm{d}x - \int_0^{x_0} F(x)w(x,t)\mathrm{d}x \\ &- \int_0^1\int_0^x G(x)l(x,\xi)w(\xi,t)\mathrm{d}\xi\mathrm{d}x \\ &- \int_0^{x_0}\int_0^x F(x)l(x,\xi)w(\xi,t)\mathrm{d}\xi\mathrm{d}x. \end{aligned} \tag{33}$$

Therefore, we have shown that the inverse transformation is well established.

V. STABILITY OF THE CLOSED-LOOP SYSTEM

To prove stability of the closed-loop system (1) with the control law (5), the stability of the target system (4) still needs ot be shown and that the inverse transformation is a bounded linear operatorneeds to be proven. We state it as the following theorem.

Theorem 2. Assume **(H)** holds, and consider the control system (1) and (5) with $k(x,y)$, $F(x)$, $G(y)$ and $\gamma(x)$ defined in (14), (17), (18) and (32) respectively. Then, for any initial condition $u(\cdot,0) \in L^2(0,1)$, the closed loop system (1) and (5) has a unique solution $(X(t), u(\cdot,t)) \in C([0,\infty), \mathbb{R}^n \times L^2(0,1))$, and there exist positive constants C and b such that

$$\|X(t)\|^2 + \|u(t)\|_2^2 \le C\left(\|X(0)\|^2 + \|u(0)\|_2^2\right)e^{-bt},$$

that is, the closed loop system (1) and (5) is exponentially stable in the sense of $(\|X(t)\|^2 + \|u(t)\|_2^2)^{\frac{1}{2}}$. Here

$$\|u(t)\|_2 := \left(\int_0^1 u^2(x,t)dx\right)^{\frac{1}{2}}$$

is the L^2 norm for space variable x, and $\|\cdot\|$ denotes the Euclidian norm of a vector.

Proof. Firstly, we show the target system (4) is exponentially stable via the Lyapunov function method. According to (2), we know $w(\cdot,0) \in L^2(0,1)$. Then, by operator semigroup theory, (4) has a unique solution $w(\cdot,t) \in C([0,+\infty), L^2(0,1))$. We define the following Lyapunov function for (4):

$$V(t) = Z(t)^T PZ(t) + \frac{a}{2}\|w(t)\|_2^2, \tag{34}$$

where the matrices $P = P^T > 0$ and $Q = Q^T > 0$ are the solution to the Lyapunov equation

$$P(A - G'(1)K) + (A - G'(1)K)^T P = -Q \tag{35}$$

and the parameter $a > 0$ chosen later.

Taking the derivative of the Lyapunov function V with respect to t, then using (4) and (35), we obtain

$$\begin{aligned}\dot V(t) =&\dot Z(t)^T PZ(t) + Z(t)^T P\dot Z(t) + a\int_0^1 w_t(x,t)wdx\\ =&Z(t)^T(P(A - G'(1)K)\\ &+ (A - G'(1)K)^T P)Z(t) + a\int_0^1 w_{xx}(x,t)wdx\\ =&-Z(t)^T QZ(t) - a\|w_x(t)\|_2^2.\end{aligned}$$

By the Poincaré's Inequality and $w(1,t) = 0$, we have

$$\begin{aligned}\|w(t)\|_2 &= \left(\int_0^1 w^2(x,t)dx\right)^{\frac{1}{2}}\\ &\le \left(2w^2(1,t) + 4\int_0^1 w_x^2(x,t)dx\right)^{\frac{1}{2}}\\ &= 2\|w_x(t)\|_2.\end{aligned} \tag{36}$$

Therefore, from (36), we have

$$\dot V(t) \le -b\left(\|Z(t)\|^2 + \|w(t)\|_2^2\right) = -bV(t),$$

where

$$b := \min\left\{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, \frac{a}{4}\right\},$$

and thus

$$V(t) \le V(0)e^{-bt}.$$

Secondly, we establish the relation between the norm $(\|X(t)\|^2 + \|u(t)\|_2^2)^{\frac{1}{2}}$ and $V(t)$. From the transformations (2) and (3), it holds that

$$\begin{aligned}\|w(t)\|_2 \le& \|u(t)\|_2 + \|\gamma(x)Z(t)\|_2\\ &+ \left\|\int_0^x k(x,y)u(y,t)dy\right\|_2,\end{aligned} \tag{37}$$

$$\begin{aligned}\|Z(t)\| \le& \|X(t)\| + \left\|\int_0^1 G(y)u(y,t)dy\right\|\\ &+ \left\|\int_0^{x_0} F(y)u(y,t)dy\right\|.\end{aligned} \tag{38}$$

By Cauchy-Schwarz's Inequality, we have

$$\begin{aligned}\|\gamma(x)Z(t)\|_2^2 &= \int_0^1 \|\gamma(x)Z(t)\|^2dx\\ &\le \int_0^1 \|\gamma(x)\|^2\ \|Z(t)\|^2dx\\ &= \|\gamma\|_2^2\|Z(t)\|^2\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_0^x k(x,y)u(y,t)dy\right\|_2^2\\ =&\int_0^1\left(\int_0^x k(x,y)u(y,t)dy\right)^2dx\\ \le&\int_0^1\left(\int_0^x k^2(x,y)dy\int_0^x u^2(y,t)dy\right)dx\\ \le&\int_0^1\int_0^1 k^2(x,y)dydx\int_0^1\left(\int_0^1 u^2(y,t)dy\right)dx\\ =&\kappa^2\|u(t)\|_2^2,\end{aligned}$$

where

$$\kappa := \left(\int_0^1\int_0^1 k^2(x,y)dydx\right)^{\frac{1}{2}}.$$

Therefore, we have

$$\|w(t)\|_2 \le (1+\kappa)\|u(t)\|_2 + \|\gamma\|_2\|Z(t)\|. \tag{39}$$

From the Cauchy-Schwarz's Inequality,

$$\begin{aligned}\left\|\int_0^1 G(y)u(y,t)dy\right\| \le& \left(\int_0^1 \|G(y)\|^2dy\int_0^1 u^2(y,t)dy\right)^{\frac{1}{2}}\\ =&\mu\|u(t)\|_2,\end{aligned}$$

where

$$\mu := \left(\int_0^1 \|G(y)\|^2dy\right)^{\frac{1}{2}}.$$

Similarly, we also obtain

$$\begin{aligned}\left\|\int_0^{x_0} F(y)u(y,t)dy\right\| &\le \left(\int_0^1 \|F(y)\|^2dy\int_0^1 u^2dy\right)^{\frac{1}{2}}\\ &= \alpha\|u(t)\|_2,\end{aligned}$$

where

$$\alpha := \left(\int_0^1 \|F(y)\|^2dy\right)^{\frac{1}{2}}.$$

Then,

$$\|Z(t)\| \le \|X(t)\| + (\mu+\alpha)\|u(t)\|_2. \tag{40}$$

Adopting the above similar procedure, according to (23) and (33), we have

$$\begin{aligned}\|X(t)\| \le & (1+\mu\eta+\alpha\eta)\|Z(t)\| \\ & + (\iota+1)(\mu+\alpha)\|w(t)\|_2,\end{aligned} \tag{41}$$

$$\|u(t)\|_2 \le (1+\iota)\|w(t)\|_2 + \eta\|Z(t)\|, \tag{42}$$

where

$$\eta := \left(\int_0^1 \|\varphi(y)\|^2 dy\right)^{\frac{1}{2}},$$

$$\iota := \left(\int_0^1\int_0^1 l^2(x,y)dydx\right)^{\frac{1}{2}}.$$

In terms of (41) and (42), we obtain

$$\begin{aligned}&\|X(t)\|^2 + \|u(t)\|_2^2 \\ \le&\big((1+\mu\eta+\alpha\eta)\|Z(t)\| + (\iota+1)(\mu+\alpha)\|w(t)\|_2\big)^2 \\ &+ \big((1+\iota)\|w(t)\|_2 + \eta\|Z(t)\|\big)^2 \\ \le&2\big(\eta^2 + (1+\mu\eta+\alpha\eta)^2\big)\|Z(t)\|^2 \\ &+ 2\big((1+\iota)^2((\mu+\alpha)^2+1)\big)\|w(t)\|_2^2 \\ \le&\frac{2\big(\eta^2+(1+\mu\eta+\alpha\eta)^2\big)}{\lambda_{\min}(P)}Z(t)^T PZ(t) \\ &+ \frac{4\big((1+\iota)^2((\mu+\alpha)^2+1)\big)}{a}\frac{a}{2}\|w(t)\|_2^2 \\ \le&\beta\left(Z(t)^T PZ(t) + \frac{a}{2}\|w(t)\|_2^2\right) \\ =&\beta V(t) \le \beta V(0)e^{-bt},\end{aligned} \tag{43}$$

where

$$\begin{aligned}\beta := \max\Bigg\{&\frac{2\big(\eta^2+(1+\mu\eta+\alpha\eta)^2\big)}{\lambda_{\min}(P)}, \\ &\frac{4\big((1+\iota)^2((\mu+\alpha)^2+1)\big)}{a}\Bigg\}.\end{aligned} \tag{44}$$

By (39), (40), (41) and (42), there exists a constant C such that

$$\|X(t)\|^2 + \|u(t)\|_2^2 \le C\big(\|X(0)\|^2 + \|u(0)\|_2^2\big)e^{-bt}.$$

Finally, combining with the existence of the inverse transformation (23), we conclude that the closed-loop system (1) with control law (5) has a unique weak solution $(X(t), u(\cdot,t)) \in C([0,\infty), \mathbb{R}^n \times L^2(0,1))$, which finishes the proof of Theorem 2.

VI. SIMULATION

We give the simulation for system (1) under the controller

$$U(t) = KZ(t) + \int_0^1 k(1,y)u(y,t)dy$$

with special parameters and initial data. Figs 1 and 2 show the behavior of states $(X(t), u(x,t))$ of closed-loop system (1) and (5) with parameters $x_0 = 0.5$, $\lambda = 0$, $A = 1$, $B_0 = 1$ and $B = 2$. We see from the figures that the closed-loop system is stable, which shows that the controller design stabilizes the plant successfully.

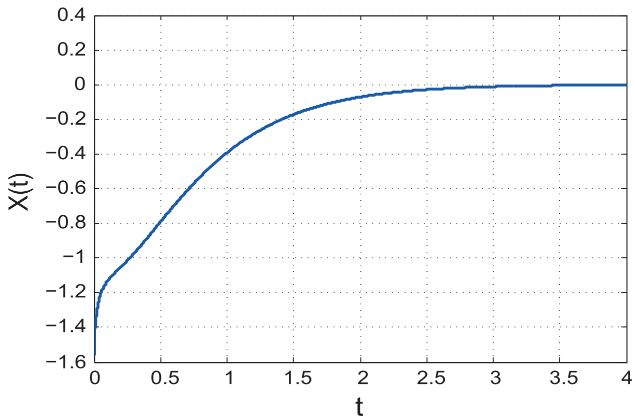

Fig. 1. The response of closed-loop system state $X(t)$ with given parameters. [Color figure can be viewed at wileyonlinelibrary.com]

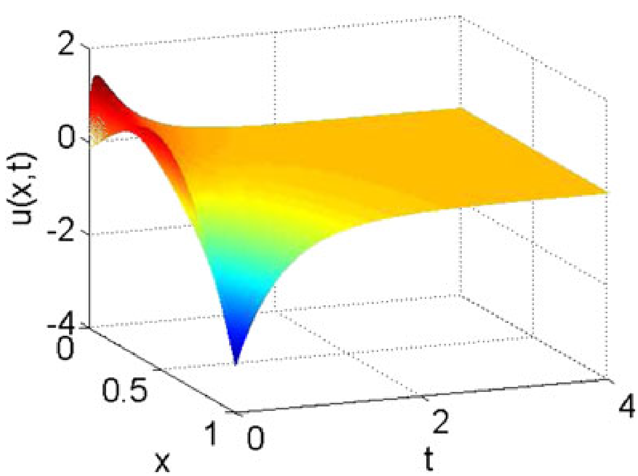

Fig. 2. The response of closed-loop system state $u(x,t)$ with given parameters. [Color figure can be viewed at wileyonlinelibrary.com]

VII. CONCLUSION AND FURTHER WORKS

We considered the exponential stabilization for a heat-ODE system (1) cascaded at a boundary point and an intermediate point via a novel forward and inverse transformation. Meanwhile, we showed that the kernel functions in the transformations are of the class C^2. In fact, for the system (1) cascaded with multiple points, its stabilization can also be obtained using a similar procedure as the one in this paper. However, for a general heat-ODE system

$$\begin{cases} \dot{X}(t) = AX(t) + \int_0^1 B(y)u(y,t)dy, \\ u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u_x(0,t) = 0, \ u(1,t) = U(t), \\ u(x,0) = u_0(x), \\ X(0) = X_0, \end{cases} \tag{45}$$

where $A \in \mathbb{R}^{n\times n}$, $B(\cdot) \in H^{-1}((0,1);\mathbb{R}^{n\times 1})$, λ is a constant. Under the assumption $B(\cdot) \neq 0$, how to propose the stabilizing controller design is a challenging work, because the procedure of this paper doesn't work any more for (45).

REFERENCES

1. Coron, J. M., *Control and Nonlinearity*, American Mathematical Society, Providence, RI (2007).
2. Liu, W., *Elementary Feedback Stabilization of the Linear Reaction-convection-diffusion Equation and the Wave Equations*, Mathématiques & Applications, Vol. 66, Springer-Verlag, Berlin (2010).
3. Krstic, M. and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, SIAM, Philadelphia PA (2008).
4. Boskovic, D. M., A. Balogh, and M. Krstic, "Backstepping in infinite dimension for a class of parabolic distributed parameter systems," *Math. Control Signal Syst.*, Vol. 16, No. 1, pp. 44–75 (2003).
5. Guo, C., C. Xie, and C. Zhou, "Stabilization of a spatially non-causal reaction-diffusion equation by boundary control," *Int. J. Robust Nonlinear Control*, Vol. 24, No. 1, pp. 1–17 (2014).
6. Zhou, Z. and C. Guo, "Stabilization of linear heat equation with a heat source at intermediate point by boundary control," *Automatica*, Vol. 49, No. 2, pp. 448–456 (2013).
7. Bekiaris-Liberis, N. and M. Krstic, "Compensating the distributed effect of diffusion and counter-convection in multi-input and multi-output LTI systems," *IEEE Trans. Autom. Control*, Vol. 56, No. 3, pp. 637–643 (2011).
8. Krstic, M., "Compensating actuator and sensor dynamics governed by diffusion PDEs," *Syst. Control Lett.*, Vol. 58, No. 5, pp. 372–377 (2009).
9. Susto, G. A. and M. Krstic, "Control of PDE-ODE cascades with Neumann interconnections," *J. Frankl. Inst. Eng. Appl. Math.*, Vol. 347, No. 1, pp. 284–314 (2010).
10. Zhou, Z. and C. Xu, "Stabilization of a second order ODE-heat system coupling at intermediate point," *Automatica*, Vol. 60, pp. 57–64 (2015).
11. Tang, S. and C. Xie, "Stabilization for a coupled PDE-ODE control system," *J. Frankl. Inst. Eng. Appl. Math.*, Vol. 348, No. 8, pp. 2142–2155 (2011).
12. Tang, S. and C. Xie, "State and output feedback boundary control for a coupled PDE-ODE system," *Syst. Control Lett.*, Vol. 60, No. 8, pp. 540–545 (2011).
13. Zhou, Z. and S. Tang, "Boundary stabilization of a coupled wave-ODE system with internal anti-damping," *Int. J. Control*, Vol. 85, No. 11, pp. 1683–1693 (2012).

VIII. APPENDIX

We will show the coherence for the controllability assumption in Theorem 2 with the one in [7] for a special case with $\lambda = 0$. Indeed, if $\lambda = 0$, the system (1) degenerates to the original system with $B(x) = B_0\delta(x) + B\delta(x-x_0)$ and $b = 0$ in [7]. Then, the kernel $g(x)$ in [7] satisfies

$$\begin{cases} g''(x) = Ag(x) - B(x), \\ g(1) = 0, g'(0) = 0, \end{cases} \tag{46}$$

for which the solution is

$$\begin{aligned} g(x) = &\begin{pmatrix} I_n & 0 \end{pmatrix} e^{Dx} \begin{pmatrix} I_n \\ 0 \end{pmatrix} E^{-1} \\ &\times \int_0^1 \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(1-y)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B(y)dy \\ &- \begin{pmatrix} I_n & 0 \end{pmatrix} \int_0^x e^{D(x-y)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B(y)dy, \end{aligned}$$

where

$$D = \begin{pmatrix} 0 & I_n \\ A & 0 \end{pmatrix}, E = \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D} \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \tag{47}$$

Thus,

$$
\begin{aligned}
&g'(1)\\
&= \begin{pmatrix} I_n & 0 \end{pmatrix} e^D \begin{pmatrix} 0 \\ A \end{pmatrix} E^{-1}\\
&\quad \times \int_0^1 \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(1-y)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B(y)dy\\
&\quad - \begin{pmatrix} I_n & 0 \end{pmatrix} \int_0^1 e^{D(1-y)} \begin{pmatrix} I_n \\ 0 \end{pmatrix} B(y)dy\\
&= \begin{pmatrix} I_n & 0 \end{pmatrix} e^D\\
&\quad \times \begin{pmatrix} 0 \\ AE^{-1}\int_0^1 \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(1-y)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B(y)dy \end{pmatrix}\\
&\quad - \begin{pmatrix} I_n & 0 \end{pmatrix} \int_0^1 e^{D(1-y)} \begin{pmatrix} I_n \\ 0 \end{pmatrix} B(y)dy.
\end{aligned}
$$

We know

$$
e^D := \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}, e^{-Dx_0} := \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}
$$

with block matrices

$$
\begin{aligned}
&I_{11} = I_{22} = \sum_{n=0}^{+\infty} \frac{A^n}{(2n)!},\ I_{12} = \sum_{n=0}^{+\infty} \frac{A^n}{(2n+1)!},\\
&I_{21} = \sum_{n=0}^{+\infty} \frac{A^{n+1}}{(2n+1)!} = AI_{12}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&J_{11} = J_{22} = \sum_{n=0}^{+\infty} \frac{(Ax_0^2)^n}{(2n)!},\ J_{12} = -\sum_{n=0}^{+\infty} \frac{A^n x_0^{2n+1}}{(2n+1)!},\\
&J_{21} = -\sum_{n=0}^{+\infty} \frac{A^{n+1} x_0^{2n+1}}{(2n+1)!} = AJ_{12}.
\end{aligned}
$$

According to (47), we have $E = I_{11}$. Also, we know

$$
\begin{aligned}
&\int_0^1 \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(1-y)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B(y)dy\\
&= \begin{pmatrix} I_n & 0 \end{pmatrix} e^D \begin{pmatrix} 0 \\ I_n \end{pmatrix} B_0\\
&\quad + \begin{pmatrix} I_n & 0 \end{pmatrix} e^{D(1-x_0)} \begin{pmatrix} 0 \\ I_n \end{pmatrix} B\\
&= \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} \begin{pmatrix} 0 \\ B_0 \end{pmatrix} + \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} \begin{pmatrix} J_{12}B \\ J_{22}B \end{pmatrix}\\
&= I_{12}B_0 + (I_{11}J_{12} + I_{12}J_{22})B.
\end{aligned} \tag{48}
$$

Finally, simplifying the expression $g'(1)$, we obtain

$$
\begin{aligned}
&g'(1)\\
&= \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} \begin{pmatrix} 0 \\ AI_{11}^{-1}(I_{12}B_0 + (I_{11}J_{12} + I_{12}J_{22})B) \end{pmatrix}\\
&\quad - \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} \begin{pmatrix} B_0 \\ 0 \end{pmatrix} - \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} e^{-Dx_0} \begin{pmatrix} B \\ 0 \end{pmatrix}\\
&= I_{12}AI_{11}^{-1}(I_{12}B_0 + (I_{11}J_{12} + I_{12}J_{22})B)\\
&\quad - I_{11}B_0 - I_{11}J_{11}B - I_{12}J_{21}B.
\end{aligned}
$$

Next, we compute $G'(1)$. According to (11), (18), (19) and (20), we obtain

$$
\begin{aligned}
&G'(1)\\
&= \begin{pmatrix} I_n & 0 \end{pmatrix} e^D\\
&\quad \times \begin{pmatrix} -\begin{pmatrix} 0 & I_n \end{pmatrix} e^{-Dx_0} \begin{pmatrix} 0 \\ B \end{pmatrix} - B_0 \\ AE^{-1} \begin{pmatrix} I_n & 0 \end{pmatrix} e^D \begin{pmatrix} 0 \\ B_0 + F'(0) \end{pmatrix} \end{pmatrix}\\
&= \begin{pmatrix} I_{11} & I_{12} \end{pmatrix}\\
&\quad \times \begin{pmatrix} -J_{22}B - B_0 \\ AE^{-1} \begin{pmatrix} I_{11} & I_{12} \end{pmatrix} \begin{pmatrix} 0 \\ J_{22}B + B_0 \end{pmatrix} \end{pmatrix}\\
&= -I_{11}J_{22}B - I_{11}B_0 + I_{12}AE^{-1}(I_{12}B_0 + I_{12}J_{22})B).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g'(1) - G'(1) &= I_{12}AE^{-1}I_{11}J_{12}B - I_{12}J_{21}B\\
&= I_{12}AI_{11}^{-1}I_{11}J_{12}B - I_{12}J_{21}B\\
&= I_{12}AJ_{12}B - I_{12}J_{21}B\\
&= I_{12}J_{21}B - I_{12}J_{21}B\\
&= 0,
\end{aligned} \tag{49}
$$

which shows the coherence for the controllability assumption of $(A, G'(1))$ in Theorem 2 and $(A, g'(1))$ in [7] for a special case with $\lambda = 0$.

Zhiyuan Zhen received his B.Sc. in Information and Computing Science from China Huang Huai University, China, in 2013 and his M.Sc. in Operational Research and Cybernetics from Southwest University, China, in 2016. He is studying for a Ph.D. at Hunan University, China. His research interests focus on control of the supply chains and distributed parameter systems.

Shu-Xia Tang received her Ph.D. in Mechanical Engineering in 2016 from Mechanical & Aerospace Engineering, University of California, San Diego, USA. She is currently a postdoctoral research fellow at the Department of Applied Mathematics, University of Waterloo, Canada. Her main research interests are stability analysis and control design of distributed parameter systems. Moreover, lithium-ion battery applications and multi-agent cooperative control also fall into her research interests.

Zhongcheng Zhou received his B.Sc. and M.Sc. in Mathematics from Northeast Normal University, China, in 2001 and 2004 respectively, his Ph.D. from the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences in 2009. Now, he is working at Southwest University as an associate professor. His research interests focus on distributed parameter systems control.