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Stabilization of a Coupled PDE-ODE System by Boundary Control

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***Abstract*— A coupled system of an ODE and a diffusion PDE is considered in this paper. Special techniques as well as the method of PDE backstepping are employed to construct controllers. Through transforming the system into an exponentially stable PDE-ODE cascade, a state feedback boundary controller is established. Moreover, an observer for anti-collocated setup is proposed, and the observer error is shown to exponentially converge to zero, then an output feedback boundary controller is obtained. For a scalar coupled PDE-ODE system, the boundary controller and observer, as well as the solution of the closed-loop system are given explicitly.**

***Index Terms*— Coupled PDE-ODE system, Boundary control, Output feedback**

I. Introduction

Coupling takes place in many aspects such as electromagnetic coupling, mechanical coupling, and coupled chemical reactions.

Controllability of coupled PDE-PDE systems have been studied in [1], [2], [7]–[9], [12]–[18]. Designing of boundary controllers and observers for coupled PDE-PDE systems as well as coupled PDE-ODE systems, however, is an original area.

The system to be studied in this paper couples an ODE with a heat equation, where the interconnection between the PDE and the ODE is two-directional, that is, the ODE acts back on PDE at the same time as the PDE acts on the ODE. Since the overall coupled system is more complicated than just a single ODE or a single PDE, and even more complicated than a PDE-ODE cascade, difficulties occur.

The most intuitive method to tackle coupling in the system is resorting to decouple it. But this is not practicable for all the time. The method of backstepping can be recurred to here, which has been used in designing of boundary feedback controllers of cascaded PDE-ODE systems in [3]–[6], [10], [11], where the interconnection between the PDE and the ODE is one-directional. Still, some other special techniques are also used in solving the problems.

This paper is organized as follows. In Section II the problem is formulated and analyzed. In Section III a state feedback boundary controller is designed to stabilize the coupled PDE-ODE system. In Section IV an observer is designed, and the output feedback boundary control problem is solved. An example is given in Section V, where the controller and observer for a scalar coupled PDE-ODE system, as well as solutions to the closed-loop systems, are given explicitly. In Section VI, some conclusions and comments are made on stabilization of the coupled systems.

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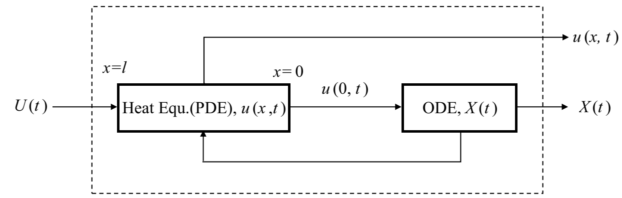

Fig. 1. The coupled system of the heat equation PDE and the ODE

II. Problem formulation and analysis

The following model which couples a finite-dimensional system of ODE with a heat equation of PDE

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{1}$$
$$u_t(x,t) = u_{xx}(x,t) + CX(t),\ x \in (0,l) \tag{2}$$
$$u_x(0,t) = 0 \tag{3}$$
$$u(l,t) = U(t) \tag{4}$$

is to be considered, where $X(t) \in \mathbb{R}^n$ is the ODE state, and the pair (A, B) is assumed to be stabilizable; $u(x,t) \in \mathbb{R}$ is the PDE state, and C^T is a constant vector; $U(t)$ is the scalar input to the entire system. The coupled system is depicted in Fig. 1. The control objective is to exponentially stabilize the system signals $u(x,t)$ and $X(t)$.

The solution to the ODE (1) can be represented by

$$X(t) = X(0)e^{At} + \int_0^t e^{A(t-\tau)}Bu(0,\tau)d\tau$$

Substituting the solution into (2), the following non-coupled PDE system

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t) \\ &+ C\left(X(0)e^{At} + \int_0^t e^{A(t-\tau)}Bu(0,\tau)d\tau\right) \\ u_x(0,t) =& 0 \\ u(l,t) =& U(t) \end{aligned}$$

is obtained. Intuitively, this system is stabilizable.

However, to achieve the stabilization of the system $(1)-(4)$ in a strict manner, compared with doing the decoupling directly, PDE backstepping is more effective.

The method of PDE backstepping is to seek an invertible transformation $(X,u) \mapsto (X,w)$ to convert the system

$(1)-(4)$ into an exponentially stable target system, e.g., the following system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{5}$$
$$w_t(x,t) = w_{xx}(x,t) \tag{6}$$
$$w_x(0,t) = 0 \tag{7}$$
$$w(l,t) = 0 \tag{8}$$

where K is chosen such that $A+BK$ is Hurwitz. Stabilization of the above target system can be proved by following almost the same line as the correspondent proof in [4] except that the parameters $\underline{\delta}$ and $\overline{\delta}$ are chosen as

$$\underline{\delta} = \frac{\min\{\frac{a}{2}, \lambda_{\min}(P)\}}{\max\{\beta_1, \beta_2+1\}}$$
$$\overline{\delta} = \max\left\{\frac{a}{2}\alpha_1, \frac{a}{2}\alpha_2 + \lambda_{\max}(P)\right\}$$

Thus, with the invertibility of the transformation $(X,u) \mapsto (X,w)$, exponential stability of the resulting closed-loop system will be achieved.

III. State feedback controller design

A. Stabilization by state feedback

The transformation $(X,u) \mapsto (X,w)$ is postulated in the following form

$$w(x,t) = u(x,t) - \int_0^x \kappa(x,y)u(y,t)dy - M(x)X(t) \tag{9}$$

where the gain functions $\kappa(x,y) \in \mathbb{R}$ and $M(x)^T \in \mathbb{R}^n$ are to be determined.

By matching the systems $(1)-(3)$ and $(5)-(7)$, it can be obtained that the desired kernel functions $\kappa(x,y)$ and $M(x)$ satisfy the following conditions

$$\kappa_{xx}(x,y) = \kappa_{yy}(x,y) \tag{10}$$
$$\kappa(x,x) = 0 \tag{11}$$
$$\kappa_y(x,0) = -M(x)B \tag{12}$$

and

$$M''(x) - M(x)A - \int_0^x \kappa(x,y)dyC + C = 0 \tag{13}$$
$$M(0) = K \tag{14}$$
$$M'(0) = 0 \tag{15}$$

What must be emphasized here is that the PDE $(10)-(12)$ and ODE $(13)-(15)$ are weakly coupled, which can be decoupled by using some techniques.

Firstly, the solution to the PDE $(10)-(12)$ can be obtained as

$$\kappa(x,y) = \int_0^{x-y} M(\sigma)Bd\sigma \tag{16}$$

Let

$$m(s) = \int_0^s M(\sigma)Bd\sigma$$

then

$$\kappa(x,y) = m(x-y)$$

Substituting (16) into (13), it is obtained that

$$M''(x) - M(x)A - \int_0^x \int_0^{x-y} M(\sigma)Bd\sigma dyC + C = 0$$

which is a non-homogeneous linear ODE of second order. Changing the order of integration and differentiating the ODE twice, the following four order ODE

$$M^{(4)}(x) - M''(x)A - M(x)BC = 0 \tag{17}$$

and initial values

$$M''(0) = KA - C,\ M^{(3)}(0) = 0$$

are obtained.

Let

$$\Gamma(x) = \begin{pmatrix} M(x) & M'(x) & M''(x) & M^{(3)}(x) \end{pmatrix}$$

and I be a unit matrix, then (17) is written into

$$\Gamma'(x) = \Gamma(x)D$$

where

$$D = \begin{pmatrix} 0 & 0 & 0 & BC \\ I & 0 & 0 & 0 \\ 0 & I & 0 & A \\ 0 & 0 & I & 0 \end{pmatrix}$$

Hence, the solution to the ODE $(13)-(15)$ is

$$M(x) = \Gamma(0)e^{Dx}E$$

where

$$\Gamma(0) = \begin{pmatrix} K & 0 & KA-C & 0 \end{pmatrix},\ E = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The inverse transformation $(X,w) \mapsto (X,u)$ is postulated in the following form

$$u(x,t) = w(x,t) + \int_0^x \iota(x,y)w(y,t)dy + N(x)X(t) \tag{18}$$

where the kernel functions $\iota(x,y) \in \mathbb{R}$ and $N(x)^T \in \mathbb{R}^n$ can be obtained as

$$N(x) = \left(K - C(A+BK)^{-1}\right)F(x) + C(A+BK)^{-1}$$
$$\iota(x,y) = \int_0^{x-y} N(\sigma)Bd\sigma = n(x-y)$$

where

$$F(x) = \begin{pmatrix} I & 0 \end{pmatrix} e^{\Delta x} \begin{pmatrix} I \\ 0 \end{pmatrix}$$
$$n(s) = \int_0^s N(\sigma)Bd\sigma$$

and

$$\Delta = \begin{pmatrix} 0 & A+BK \\ I & 0 \end{pmatrix}$$

By evaluating (9) at $x = l$ and from the boundary conditions (4) and (8), the controller

$$U(t) = \int_0^l m(l-y)u(y,t)dy + M(l)X(t) \tag{19}$$

is obtained.

Furthermore, the solution to the system $(1)-(4), (19)$ can also be obtained. Firstly, the heat equation $(6)-(8)$ is solved, and the solution

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t}\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\cdot\Phi \tag{20}$$

is obtained, where

$$\Phi = \int_0^1 w_0(\xi)\cos\left(\left(n+\frac{1}{2}\right)\pi\xi\right)d\xi$$

and the initial condition $w_0(x)$ can be calculated via (9). Then, the solution to the closed-loop system $(1)-(4), (19)$ can be obtained from

$$X(t) = X(0)e^{(A+BK)t} + \int_0^t e^{(A+BK)(t-\tau)}Bw(0,\tau)d\tau \tag{21}$$

and (18).

Through the results established, the following theorem can be shown.

Theorem 1: For any initial data $X(0) \in \mathbb{R}, u(\cdot,0) \in L^2[0,l]$, the closed-loop system consisting of the plant $(1)-(4)$ and the control law (19) is exponentially stabilized in the sense of the norm

$$\|\,(u(\cdot,t), X(t))\,\|^2 = \int_0^l u(x,t)^2dx + \|X(t)\|^2$$

where $\|X(t)\|$ denotes the Euclidian norm.

IV. Observer design and output feedback

To implement the control law (19), the information of the signal $u(x,t)$ is supposed to be measurable. Sometimes, the information is measurable only at one of the ends, or for economic considerations, is measured only at one end. In this situation, an observer is necessary to track the signal $u(x,t)$. Consider the case that only $u(0,t)$ is available for measurement. Since the input is at the opposite end ($x = l$), it is called observer for anti-collocated setup.

A. Observer design for anti-collocated setup

Observer with Dirichlet actuation of the following form

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(0,t) + P_0\left(u(0,t) - \hat{u}(0,t)\right) \tag{22}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + C\hat{X}(t) + p_1(x)\left(u(0,t) - \hat{u}(0,t)\right) \tag{23}$$

$$\hat{u}_x(0,t) = p_2\left(u(0,t) - \hat{u}(0,t)\right) \tag{24}$$

$$\hat{u}(l,t) = U(t) \tag{25}$$

is to be designed to achieve exponential stabilization of error system, where P_0 is a constant vector, $p_1(x)$ is a function, p_2 is a constant. Write the observer error as

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$$
$$\tilde{X}(t) = X(t) - \hat{X}(t)$$

then the error system

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}(0,t) \tag{26}$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + C\tilde{X}(t) - p_1(x)\tilde{u}(0,t) \tag{27}$$

$$\tilde{u}_x(0,t) = -p_2\tilde{u}(0,t) \tag{28}$$

$$\tilde{u}(l,t) = 0 \tag{29}$$

is obtained.

A transformation of the form

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \Theta(x)\tilde{X}(t) \tag{30}$$

is also to be looked for to convert the system $(26)-(29)$ into a stable target system

$$\dot{\tilde{X}}(t) = \left(A - P_0\Theta(0)\right)\tilde{X}(t) - P_0\tilde{w}(0,t) \tag{31}$$

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) \tag{32}$$

$$\tilde{w}_x(0,t) = 0 \tag{33}$$

$$\tilde{w}(l,t) = 0 \tag{34}$$

To determine the transformation, $\Theta(x)$, along with output injection functions $P_0, p_1(x)$ and p_2 are to be determined.

A necessary and sufficient condition for $(31)-(34)$ to hold is that

$$\Theta''(x) - \Theta(x)A + C = 0 \tag{35}$$

$$\Theta'(0) = 0 \tag{36}$$

$$\Theta(l) = 0 \tag{37}$$

and

$$p_1(x) = \Theta(x)P_0 \tag{38}$$

$$p_2 = 0 \tag{39}$$

To construct the solution to ODE $(35)-(37)$, a lemma is shown firstly.

Lemma 1: Let

$$F = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix},\ G = \begin{pmatrix} I & 0 \end{pmatrix} e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

then G is a singular matrix if and only if A has an eigenvalue $(2k+1)^2\pi^2/l^2, k \in \mathbb{N}$.

When A has no eigenvalue as $(2k+1)^2\pi^2/l^2, k \in \mathbb{N}$, the solution to the non-homogeneous linear ODE two-point-boundary-value problem $(35)-(37)$ is as

$$\Theta(x) = \Upsilon(x)e^{Fx}\begin{pmatrix} I \\ 0 \end{pmatrix} \tag{40}$$

where

$$\Upsilon(x) = \begin{pmatrix} \Theta(0) & 0 \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & C \end{pmatrix} e^{-F\xi}d\xi$$

$$\Theta(0) = \int_0^l \begin{pmatrix} 0 & C \end{pmatrix} e^{-F\xi}d\xi \cdot e^{Fl}\begin{pmatrix} I \\ 0 \end{pmatrix} G^{-1}$$

Lastly, P_0 is chosen such that $A - P_0\Theta(0)$ is Hurwitz, then, all the quantities needed to implement the observer $(22) - (25)$ are determined.

The system $(31) - (34)$ is a cascade of the exponentially stable heat equation $(32) - (34)$ and the exponentially stable ODE (31). Thus, the entire observer error system is exponentially stable.

Theorem 2: Assume A has no eigenvalue as $(2k+1)^2\pi^2/l^2$, $k \in \mathbb{N}$, the observer $(22) - (25)$, with gains defined through $(38) - (40)$, guarantees that observer error exponentially converges to zero, that is, $\hat{X}(t)$ and $\hat{u}(t)$ exponentially track $X(t)$ and $u(t)$ in the sense of the norm

$$\| \left(\tilde{w}(\cdot,t), \tilde{X}(t)\right) \|^2 = \int_0^l \tilde{w}(x,t)^2 dx + \|\tilde{X}(t)\|^2$$

Proof: From the transformation (30), the following relations

$$\|\tilde{w}\|^2 \leq 2\|\tilde{u}\|^2 + 2\|\Theta\|^2|\tilde{X}|^2$$
$$\|\tilde{u}\|^2 \leq 2\|w\|^2 + 2\|\Theta\|^2|\tilde{X}|^2$$

are obtained. With a Lyapunov function

$$\tilde{V} = \tilde{X}^T\tilde{P}\tilde{X} + \frac{\tilde{a}}{2}\int_0^l \tilde{w}(x)^2 dx$$

where $\tilde{P} = \tilde{P}^T > 0$ is the solution to the Lyapunov equation

$$\tilde{P}\left(A - P_0\Theta(0)\right) + \left(A - P_0\Theta(0)\right)^T \tilde{P} = -\tilde{Q}$$

for some $\tilde{Q} = \tilde{Q}^T > 0$, it can be obtained that

$$\underline{\varrho}\left(\|\tilde{w}(t)\|^2 + |\tilde{X}(t)|^2\right) \leq \tilde{V} \leq \overline{\varrho}\left(\|\tilde{w}(t)\|^2 + |\tilde{X}(t)|^2\right)$$

where

$$\underline{\varrho} = \frac{\min\{\frac{\tilde{a}}{2}, \lambda_{\min}(\tilde{P})\}}{\max\{2, 2\|\Theta\|^2+1\}}$$
$$\overline{\varrho} = \max\{\tilde{a}, \tilde{a}\|\Theta\|^2 + \lambda_{\max}(\tilde{P})\}$$

and

$$\begin{aligned}\dot{\tilde{V}} &= -\tilde{X}^T\tilde{Q}\tilde{X} - 2\tilde{X}^T\tilde{P}P_0\tilde{w}(0,t) - \tilde{a}\|\tilde{w}_x\|^2 \\ &\leq -\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2 + 2\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(Q)}\tilde{w}(0,t)^2 - \tilde{a}\|\tilde{w}_x\|^2 \\ &\leq -\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2 - (\tilde{a} - 8\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(Q)})\|\tilde{w}_x\|^2\end{aligned}$$

where the last line is obtained by using Agmon's inequality. Take

$$\tilde{a} > 8\frac{|\tilde{P}P_0|^2}{\lambda_{\min}(\tilde{Q})}$$

and use Poincaré inequality, then

$$\dot{\tilde{V}} \leq -\tilde{b}\tilde{V}$$

where

$$\tilde{b} = \min\left\{\frac{\lambda_{\min}(\tilde{Q})}{2\lambda_{\max}(\tilde{Q})}, \frac{1}{2} - \frac{4|\tilde{P}P_0|^2}{\tilde{a}\lambda_{\min}(\tilde{Q})}\right\} > 0$$

Hence

$$\left(\|\tilde{w}(t)\|^2 + |\tilde{X}(t)|^2\right) \leq \frac{\overline{\varrho}}{\underline{\varrho}}\left(\|\tilde{w}(0)\|^2 + |\tilde{X}(0)|^2\right)e^{-\tilde{b}t}$$

for all $t \geq 0$, which means that the target system $(31) - (34)$ is exponentially stable in the sense of the norm

$$\|(\tilde{w}(\cdot,t), \tilde{X}(t))\|^2 = \int_0^l \tilde{w}(x,t)^2 dx + \|\tilde{X}(t)\|^2$$

Hence, the system $(26) - (29)$ is also exponentially stable since it is related to $(31) - (34)$ by the invertible coordinate transformation (30). ■

B. Output feedback for anti-collocated setup

Replace $u(y,t)$ with $\hat{u}(y,t)$ in (19), an output feedback control law is obtained as follows.

$$u(l,t) = \int_0^l m(l-y)\hat{u}(y,t)dy + M(l)\hat{X}(t) \tag{41}$$

Theorem 3: For any initial data $X(0), \hat{X}(0) \in \mathbb{R}$, $u(\cdot,0), \hat{u}(\cdot,0) \in L^2[0,l]$, the closed-loop the system consisting of plant $(1) - (3)$, $(22) - (25)$ and the controller (41) is exponentially stable in the sense of the norm

$$\begin{aligned}&\|(\tilde{w}(\cdot,t), \tilde{X}(t), \hat{w}(\cdot,t), \hat{X}(t))\|^2 \\ =& \int_0^l \tilde{w}(x,t)^2 dx + \|\tilde{X}(t)\|^2 + \int_0^l \hat{w}(x,t)^2 dx + \|\hat{X}(t)\|^2\end{aligned}$$

Proof: The transformation

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x m(x-y)\hat{u}(y,t)dy - M(x)\hat{X}(t) \tag{42}$$

converts $(22) - (25)$ into the system

$$\begin{aligned}\dot{\hat{X}}(t) =&(A+BK)\hat{X}(t) + B\hat{w}(0,t) \\ &+ (B+P_0)\left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)\right) \end{aligned}\tag{43}$$

$$\begin{aligned}\hat{w}_t(x,t) =&\hat{w}_{xx}(x,t) + (p_1(x) - M(x)(B+P_0) \\ &- \int_0^x m(x-y)p_1(y)dy\Big)\left(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)\right)\end{aligned}\tag{44}$$

$$\hat{w}_x(0,t) =0 \tag{45}$$

$$\hat{w}(l,t) =0 \tag{46}$$

The $(\tilde{X}, \tilde{w})$ system $(31) - (34)$ and the homogeneous part of the $(\hat{X}, \hat{w})$ system $(43) - (46)$ (without $\tilde{X}(t), \tilde{w}(0,t)$) are exponentially stable. The interconnection of the two systems $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$ is a cascade, and therefore the combined $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$ system is exponentially stable. In fact, this can also be proved by taking the weighted Lyapunov function

$$E(t) = e\tilde{V} + \hat{X}^TP\hat{X} + \frac{a}{2}\|\hat{w}\|^2$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^TP = -Q$$

for some $Q = Q^T > 0$, the constant a satisfies

$$a > \frac{8|PB|^2}{\lambda_{\min}(Q)}$$

and e is the weighting constant to be chosen later. By a lengthy calculation and using Poincaré inequality as well as Agmon's inequality, it can be proved that

$$\dot{E} \le -fE$$

for some $f > 0$.

Since the transformations (30) and (42) are invertible, exponential stability of the system $(\hat{X}, \hat{w}, \tilde{X}, \tilde{w})$ ensures exponential stability of the system $(\hat{X}, \hat{u}, \tilde{X}, \tilde{u})$. This directly implies the closed-loop stability of $(X, u, \hat{X}, \hat{u})$. ∎

V. Example

The following scalar coupled system

$$\dot{X}(t) = X(t) + u(0,t) \quad (47)$$
$$u_t(x,t) = u_{xx}(x,t) + X(t) \quad (48)$$
$$u_x(0,t) = 0 \quad (49)$$
$$u(1,t) = U(t) \quad (50)$$

with the initial conditions $u(x,0) = -5x$ and $X(0) = -5$, is to be considered.

A. State feedback controller and solutions

The feedback gain is taken as $K = -4$ in order to have $A + BK$ Hurwitz, then

$$\Gamma(0) = -\begin{pmatrix} 4 & 0 & 5 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the backstepping controller can be derived explicitly through (19), which is

$$U(t) = \Gamma(0)\int_0^1 e^{D(1-y)}u(y,t)dy \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \Gamma(0)e^D EX(t) \quad (51)$$

Thus, the resulted system is

$$\dot{X}(t) = -3X(t) + w(0,t) \quad (52)$$
$$w_t(x,t) = w_{xx}(x,t) \quad (53)$$
$$w_x(0,t) = 0 \quad (54)$$
$$w(1,t) = 0 \quad (55)$$

Furthermore, the solution to the system $(47) - (50), (51)$ is explicitly available. Firstly, the explicit solution of the heat equation $(53) - (55)$ is obtained by (20), where

$$\Phi = \frac{(n+\frac{1}{2})\pi \sin\left((n+\frac{1}{2})\pi\right)}{(n+\frac{1}{2})^4\pi^4 + (n+\frac{1}{2})^2\pi^2 - 1} \times \begin{pmatrix} 4 & 0 & 5 & 0 \end{pmatrix} e^D \begin{pmatrix} -1-(n+\frac{1}{2})^2\pi^2 \\ -3-2(n+\frac{1}{2})^2\pi^2 \\ 1 \\ 2+(n+\frac{1}{2})^2\pi^2 \end{pmatrix} + \frac{(n+\frac{1}{2})^2\pi^2 - 3}{(n+\frac{1}{2})^4\pi^4 + (n+\frac{1}{2})^2\pi^2 - 1}$$

Since

$$N(x) = -\frac{11}{3}\cosh\left(\sqrt{-3}x\right) - \frac{1}{3}$$
$$\iota(x,y) = -\frac{11}{3\sqrt{-3}}\sinh\left(\sqrt{-3}(x-y)\right) - \frac{1}{3}(x-y)$$

the solution to the closed-loop system $(47) - (50), (51)$ can finally be obtained explicitly from (21) and (18), which is

$$X(t) = -5e^{-3t} + \int_0^t e^{-3(t-\tau)}w(0,\tau)d\tau = -5e^{-3t} + 10\sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})^2\pi^2 t}\Phi\Psi_1 \quad (56)$$

$$u(x,t) = 5e^{-3t}\left(\frac{11}{3}\cosh(\sqrt{-3}x) + \frac{1}{3}\right) + 10\sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})^2\pi^2 t}\Phi\Psi_2 \quad (57)$$

where

$$\Psi_1 = \frac{e^{\left((n+\frac{1}{2})^2\pi^2-3\right)t} - 1}{(n+\frac{1}{2})^2\pi^2 - 3}$$
$$\Psi_2 = \cos\left((n+\frac{1}{2})\pi x\right) + \frac{\cos\left((n+\frac{1}{2})\pi x\right)}{3(n+\frac{1}{2})^2\pi^2} + \frac{1}{3\left((n+\frac{1}{2})^2\pi^2-3\right)}\Big(11\cos\left((n+\frac{1}{2})\pi x\right) - e^{\left((n+\frac{1}{2})^2\pi^2-3\right)t}\left(11\cosh(\sqrt{-3}x)+1\right)\Big)$$

From (56) and (57), it's evident that the closed-loop system exponentially converges to zero.

B. observer, output feedback and solutions

In this case

$$\Theta(0) = 1 - \frac{1}{\cosh 1}, \Theta(x) = 1 - \frac{\cosh x}{\cosh 1}$$

Take

$$P_0 = \frac{2\cosh 1}{\cosh 1 - 1}$$

then the backstepping observer is

$$\dot{\hat{X}}(t) = \hat{X}(t) + u(0,t) + \frac{2\cosh 1}{\cosh 1 - 1}\left(u(0,t) - \hat{u}(0,t)\right)$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \hat{X}(t) + \frac{2}{\cosh 1 - 1}(\cosh 1 - \cosh x)\left(u(0,t) - \hat{u}(0,t)\right)$$

$$\hat{u}_x(0,t) = 0$$

$$\hat{u}(1,t) = \int_0^1 m(1-y)\hat{u}(y,t)dy + M(1)\hat{X}(t)$$

Taking the observer initial conditions $\hat{u}(x,0) = 0$, $\hat{X}(0) = 0$ and following the similar steps as seeking for the solution to the closed-loop system in Section V. A, the explicit solution to the resulting error system can also be obtained

$$\tilde{X}(t) = -5e^{-t} + 10\sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})^2\pi^2 t}\tilde{\Phi}\tilde{\Psi}_1 \quad (58)$$

$$\tilde{u}(x,t) = 5\left(\frac{\cosh x}{\cosh 1} - 1\right)e^{-t} + 10\sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})^2\pi^2 t}\tilde{\Phi}\tilde{\Psi}_2 \quad (59)$$

where

$$\tilde{\Phi} = \frac{1}{(n+\frac{1}{2})^2\pi^2} - \frac{(n+\frac{1}{2})\pi\sin\left((n+\frac{1}{2})\pi\right)}{(n+\frac{1}{2})^2\pi^2 + 1}$$

$$\tilde{\Psi}_1 = 2\frac{\cosh 1\left(e^{\left((n+\frac{1}{2})^2\pi^2 - 1\right)t} - 1\right)}{(1 - \cosh 1)\left((n+\frac{1}{2})^2\pi^2 - 1\right)}$$

$$\tilde{\Psi}_2 = \cos\left((n+\frac{1}{2})\pi x\right) + 2\frac{\left(e^{\left((n+\frac{1}{2})^2\pi^2 - 1\right)t} - 1\right)(\cosh x - \cosh 1)}{(\cosh 1 - 1)\left((n+\frac{1}{2})^2\pi^2 - 1\right)}$$

And (58) and (59) markedly tell that the error system is indeed exponentially stable.

VI. Conclusions and Comments

In this paper backstepping boundary controller and observer for a coupled PDE-ODE system are developed. Meanwhile, state and output feedback boundary control problems are solved.

The method of PDE backstepping is employed here in order to decouple the system by transforming it into a PDE-ODE cascade.

There exist some difficulties in seeking for the kernel functions in this paper, since they are also coupled. Fortunately, by using some techniques, it's feasible to decouple them.

Stabilization for coupled PDE-ODE systems with boundary control is an original area with so many open problems to be considered.The more general and more complicated system

$$\dot{X}(t) = AX(t) + Bu(0,t)$$

$$u_t(x,t) = u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t) + \int_0^x d(x,y)u(y,t)dy + CX(t)$$

$$u_x(0,t) = -qu(0,t)$$

$$u_x(l,t) = U(t)$$

where $b(x), c(x), d(x,y)$ are arbitrary continuous functions, is being worked on.

More interesting areas, such as stabilization for coupled PDE-PDE systems with boundary control, are also subjects of the ongoing research.

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