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Well-posedness of networked scalar semilinear balance laws subject to nonlinear boundary control operators

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***Abstract*— Networked scalar semilinear balance laws are used as simplified macroscopic vehicular traffic models. The related initial boundary value problem is investigated, on a finite interval. The upstream boundary datum is determined by a nonlinear feedback control operator, representing the fact that traffic routing might be influenced in real time by the traffic information on the entire network. The main contribution of the present work lies in the appropriate design of nonlinear boundary control operators which meanwhile guarantee the well-posedness of the resultant systems. In detail, two different types of specific nonlinear boundary control operators are instantiated, one being Lipschitz continuous and taking into account traffic information from initial time up to present time, one using only delayed traffic information. This contribution thus presents simplified road traffic network dynamics where routing at intersections is dependent of the status of the entire network, incorporating also different classes of traffic flow.**

***Index Terms*— Semilinear balance law; boundary control; delay; routing; traffic flow**

I. Introduction

At the macroscopic scale, vehicular traffic flow systems can be modeled by networked scalar balance laws, for which the mathematical representation consists of a coupled system of nonlinear first-order partial differential equations. A precise analysis for the corresponding linearized version of these equations posed on the spatial interval $\mathbb{R}$ has been carried out in [1], [2].

Routing choices serve as essential ingredients in regulating (networked) traffic flow systems. From the control perspective, off-line routing choices are generally considered as open loop control, while most on-line routing choices can be treated as feedback control. Instead, say that we provide mathematical framework to describe the system. Networked traffic systems would ideally decide the routing policies based on the traffic density throughout the entire network. In this work, for simplicity, we consider only one intersection, although the entire theory could be generalized to more general networked traffic systems without further restrictions such as the ones where the routing at a specific intersection can also depend on traffic density on the network links not adjacent to the intersection.

We consider the system of semilinear balance laws posed on a finite spatial strip, with the conjunction being the upstream endpoint. For our analysis, we use several methods and notation in [3], [4], [5]. It is also worth mentioning that our approach of defining a broad class of routing operators and studying its well-posedness has been inspired by [6] for link dynamics based on an ordinary delay differential equation [7]. For more realistic macroscopic traffic flow models we refer to [8], [9], [10].

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II. The semilinear IBVP considered

We consider the following class of scalar semi-linear balance laws:

Definition 2.1 (The IBVP considered): Let $T \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}_{\geq 1}$. Then, we call the system

$$\begin{aligned} \boldsymbol{\rho}_t(t,x) + (\boldsymbol{\Lambda}(t,x)\boldsymbol{\rho}(t,x))_x &= \boldsymbol{f}(t,x,\boldsymbol{\rho}(t,x)), && (t,x) \in \Omega_T \\ \boldsymbol{\Lambda}(t,0)\boldsymbol{\rho}(t,0) &= \boldsymbol{u}(t) && t \in (0,T) \\ \boldsymbol{\rho}(0,x) &= \boldsymbol{\rho}_0(x) && x \in (0,1) \end{aligned}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_n)^{\mathrm{T}} : \Omega_T \to \mathbb{R}^n$ denotes the solution, $\boldsymbol{u} : (0,T) \to \mathbb{R}^n_{\geq 0}$ the **l.h.s. boundary datum**, $\boldsymbol{\rho}_0 : (0,1) \to \mathbb{R}^n_{\geq 0}$ the **initial datum**, $\boldsymbol{\Lambda} := \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ the **velocity** matrix with $\lambda_i : \Omega_T \to \mathbb{R}_{>0},\ i \in \{1, 2, \cdots, n\}$, and $\boldsymbol{f} : \Omega_T \times \mathbb{R}_{>0} \to \mathbb{R}^n$ the **semi-linearity** of the system the **i**nitial **b**oundary **v**alue **p**roblem (IBVP) posed on $\Omega_T := (0,T) \times (0,1)$.

We are interested in the well-posedness of the IBVP in Definition 2.1. In the following section, we only present the study of the well-posedness of the corresponding scalar case, keeping in mind that the well-posedness study for the system case as in Definition 2.1 is, can be obtained from the diagonal structure of the velocity in almost similar manner.

Definition 2.2 (The scalar IBVP considered): Consider the following IBVP on Ω_T:

$$\begin{aligned} \rho_t(t,x) + (\lambda(t,x)\rho(t,x))_x &= f(t,x,\rho(t,x)) && (t,x) \in \Omega_T \\ \lambda(t,0)\rho(t,0) &= u(t) && t \in (0,T) \\ \rho(0,x) &= \rho_0(x) && x \in (0,1) \end{aligned}$$

with $\rho : \Omega_T \to \mathbb{R}$ the solution, $u : (0,T) \to \mathbb{R}_{\geq 0}$ the l.h.s. boundary datum, $\rho_0 : (0,1) \to \mathbb{R}_{\geq 0}$ the initial datum, $\lambda : \Omega_T \to \mathbb{R}_{>0}$ the velocity and $f : \Omega_T \times \mathbb{R}_{>0} \to \mathbb{R}$ the semi-linearity.

We make the following assumptions on the input datum of the IBVP in Definition 2.2.

Assumption 2.1 (Velocity, initial/boundary datum, etc.): Let $T \in \mathbb{R}_{>0}$ and assume for the scalar IBVP in Definition 2.2 that

- $\rho_0 \in L^\infty((0,1); \mathbb{R}_{\geq 0})$,

- $u \in L^{\infty}((0,T);\mathbb{R}_{\geq 0})$,
- $f \in C(\Omega_T \times \mathbb{R}):\ \exists L \in \mathbb{R}_{>0}, \forall (t,x) \in \Omega_T$,
$$\forall z, y \in \mathbb{R}:\ |f(t,x,z) - f(t,x,y)| \leq L|z-y|, \quad \text{(II.1)}$$
- $\lambda \in C\left([0,T]; W^{1,\infty}((0,1);\mathbb{R}_{>0})\right)$.

Remark 2.1: The assumption of the system nonlinearity f to be global Lipschitz will be essential for the well-posedness of the IBVP.

In order to consider solutions that are not necessarily differentiable, we present the usual definition of weak solutions as follows.

Definition 2.3 (Weak solution): Given Assumption 2.1, we call a function $\rho \in C\left([0,T]; L^1(0,1)\right)$ a weak solution to the the IBVP in Definition 2.2 iff for any $\tau \in [0,T]$, $\forall \phi \in C^1\left([0,T]\times[0,1]\right)$ with $\phi(\tau,\cdot) \equiv 0$ and $\phi(\cdot,1) \equiv 0$, the following integral equality holds:

$$\begin{aligned}&\iint_{\Omega_\tau} \rho(t,x)\left(\phi_t(t,x) + \lambda(t,x)\phi_x(t,x)\right)\,\mathrm{d}t\,\mathrm{d}x\\ &+ \int_0^1 \rho_0(x)\phi(0,x)\,\mathrm{d}x + \int_0^\tau u(t)\phi(t,0)\,\mathrm{d}t\\ &+ \iint_{\Omega_\tau} f(t,x,\rho(t,x))\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0.\end{aligned}$$

We first show that this IBVP is well-posed and then present its unique solution in terms of a fixed-point equation and the method of chracteristics. It is worth noting that this fixed-point argument is commonly used in mathematical derivations, it is not circular and instead provides a classical tool to the existence proof of solutions.

Theorem 2.1 (Existence and uniqueness): Given Assumption 2.1, the IBVP in Definition 2.2 possesses a unique weak solution ρ in the sense of Definition 2.3. More precisely, the solution can be posed as the unique solution to the following fixed-point equation in ρ for $(t,x) \in \Omega_T$ almost everywhere:

$$\rho(t,x) = \begin{cases} \displaystyle\int_{\xi[t,x]^{-1}(0)}^{t} \mathcal{F}[\rho](\tau,\xi[t,x](\tau))\partial_2\xi[t,x](\tau)\,\mathrm{d}\tau \\ \quad + \frac{u\left(\xi[t,x]^{-1}(0)\right)}{\lambda(\xi[t,x]^{-1}(0),0)}\partial_2\xi[t,x](\xi[t,x]^{-1}(0)) \\ \qquad\qquad \text{for } x \leq \xi[0,0](t), \\ \displaystyle\int_0^t \mathcal{F}[\rho](\tau,\xi[t,x](\tau))\partial_2\xi[t,x](\tau)\,\mathrm{d}\tau \\ \quad + \rho_0(\xi[t,x](0))\partial_2\xi[t,x](0) \\ \qquad\qquad \text{for } x > \xi[0,0](t), \end{cases} \quad \text{(II.2)}$$

where the characteristics $\xi[t,x](\cdot)$ are defined at any time-space point $(t,x) \in \Omega_T$ by the following integral equality: for any $\tau \in [0,T]$,

$$\xi[t,x](\tau) = x + \int_t^\tau \lambda(s,\xi[t,x](s))\,\mathrm{d}s \quad \text{(II.3)}$$
$$\xi[t,x]^{-1}(0) = \tau :\Longleftrightarrow \xi[t,x](\tau) = 0, \quad x \leq \xi[0,0](t),$$

and the function $\mathcal{F}$ is defined as

$$\mathcal{F}[\rho](\cdot,*) :\equiv f(\cdot,*,\rho(\cdot,*)) \text{ on } \Omega_T. \quad \text{(II.4)}$$

In addition, the solution is nonnegative if f is nonnegative.

Proof: Given $\rho \in C([0,T];L^1(0,1))$, define a mapping $G: C([0,T];L^1(0,1)) \to C([0,T];L^1(0,1))$ for $(t,x) \in \Omega_T$ by

$$G[\rho](t,x) := \begin{cases} \int_{\xi[t,x]^{-1}(0)}^{t} \mathcal{F}[\rho](\tau,\xi[t,x](\tau))\partial_2\xi[t,x](\tau)\,\mathrm{d}\tau \\ \qquad\qquad \text{for } x \leq \xi[0,0](t), \\ \int_0^t \mathcal{F}[\rho](\tau,\xi[t,x](\tau))\partial_2\xi[t,x](\tau)\,\mathrm{d}\tau \\ \qquad\qquad \text{for } x > \xi[0,0](t). \end{cases}$$

Clearly, G is a self-mapping due to Assumption 2.1 on λ and f. So it remains to show that G is also a contraction in the Banach space $C([0,T];L^1(0,1))$. To this end, compute for $\rho^1, \rho^2 \in C([0,T];L^1((0,1)))$ and $t \in [0,T]$

$$\begin{aligned}&\|G[\rho^1](t,\cdot) - G[\rho^2](t,\cdot)\|_{L^1(0,1)}\\ &\leq \int_0^{\xi[0,0](t)} \Big| \int_{\xi[t,x]^{-1}(0)}^{t} \mathcal{F}[\rho^1](\tau,\xi[t,x](\tau))\\ &\quad - \mathcal{F}[\rho^2](\tau,\xi[t,x](\tau))) \cdot \partial_2\xi[t,x](\tau)\,\mathrm{d}\tau\Big|\,\mathrm{d}x\\ &\quad + \int_{\xi[0,0](t)}^1 \Big| \int_0^t \mathcal{F}[\rho^1](\tau,\xi[t,x](\tau))\\ &\quad - \mathcal{F}[\rho^2](\tau,\xi[t,x](\tau))) \cdot \partial_2\xi[t,x](\tau)\,\mathrm{d}\tau\Big|\,\mathrm{d}x.\end{aligned}$$

From Eq. (II.1) and Eq. (II.4), we can continue the estimate and obtain for $t \in [0,T]$,

$$\begin{aligned}&\left\|G[\rho^1](t,\cdot) - G[\rho^2](t,\cdot)\right\|_{L^1(0,1)}\\ &\leq L\int_0^{\xi[0,0](t)} \int_{\xi[t,x]^{-1}(0)}^{t} \partial_2\xi[t,x](\tau)\\ &\quad \cdot \left|\rho^1(\tau,\xi[t,x](\tau)) - \rho^2(\tau,\xi[t,x](\tau))\right|\,\mathrm{d}\tau\,\mathrm{d}x\\ &\quad + L\int_{\xi[0,0](t)}^1 \int_0^t \partial_2\xi[t,x](\tau)\\ &\quad \cdot \left|\rho^1(\tau,\xi[t,x](\tau)) - \rho^2(\tau,\xi[t,x](\tau))\right|\,\mathrm{d}\tau\,\mathrm{d}x\\ &\leq L\int_0^t \int_{\xi[\tau,0](t)}^{\xi[0,0](t)} \partial_2\xi[t,x](\tau)\\ &\quad \cdot \left|\rho^1(\tau,\xi[t,x](\tau)) - \rho^2(\tau,\xi[t,x](\tau))\right|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad + L\int_0^t \int_{\xi[0,0](t)}^1 \partial_2\xi[t,x](\tau)\\ &\quad \cdot \left|\rho^1(\tau,\xi[t,x](\tau)) - \rho^2(\tau,\xi[t,x](\tau))\right|\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \quad \text{(II.5)}$$

where we have used the positivity of $\partial_2\xi$ detailed in Remark 2.2. Finishing our estimate in Eq. (II.5), we perform an integration per substitution and obtain

$$\begin{aligned}&\|G[\rho^1](t,\cdot) - G[\rho^2](t,\cdot)\|_{L^1((0,1))}\\ &\leq L\int_0^t \left\|\rho^1(\tau,\cdot) - \rho^2(\tau,\cdot)\right\|_{L^1((0,1))}\,\mathrm{d}\tau\\ &\leq Lt\left\|\rho^1 - \rho^2\right\|_{C([0,t];L^1((0,1)))}.\end{aligned}$$

Define another self-mapping $J: C\left([0,T];L^1((0,1))\right)$ for $(t,x) \in \Omega_T$ by

$$J[\rho](t,x) := G[\rho](t,x) + \begin{cases} \lambda(\xi[t,x]^{-1}(0),0)^{-1} \cdot u\left(\xi[t,x]^{-1}(0)\right) \\ \cdot\partial_2\xi[t,x]\left(\xi[t,x]^{-1}(0)\right) & \text{for } x \le \xi[0,0](t), \\ \rho_0(\xi[t,x](0))\partial_2\xi[t,x](0) & \text{for } x > \xi[0,0](t). \end{cases}$$

Again, it is clear that J is a self-mapping. Using our previous established estimate, we obtain for any $t \in [0,T]$ that

$$\begin{aligned} &\|J[\rho^1](t,\cdot) - J[\rho^2](t,\cdot)\|_{L^1((0,1))} \\ &\le \|G[\rho^1](t,\cdot) - G[\rho^2](t,\cdot)\|_{L^1((0,1))} \\ &\le Lt\left\|\rho^1-\rho^2\right\|_{C([0,t];L^1((0,1)))}. \end{aligned}$$

By choosing $t^* \in (0,\frac{1}{2L})$, we obtain

$$\|J[\rho^1] - J[\rho^2]\|_{C([0,t^*];L^1(0,1))} \le \tfrac{1}{2}\left\|\rho^1-\rho^2\right\|_{C([0,t^*];L^1((0,1)))}$$

so that J is a contraction. From Banach's fixed-point theorem there exists a unique $\rho^* \in C\left([0,t^*];L^1(0,1)\right)$, i.e.,

$$J(\rho^*) \equiv \rho^* \text{ in } C([0,t^*];L^1((0,1))). \tag{II.6}$$

In order to prove that Eq. (II.6) is indeed a weak solution of Definition 2.2 in the sense of Definition 2.3, the following identity which is a direct consequence of the previously established fixed-point solution in Eq. (II.6) is used:

$$\begin{aligned} &\iint_{[0,\tau]\times[0,1]} \rho^*(t,x)\left(\phi_t(t,x) + \lambda(t,x)\phi_x(t,x)\right)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\tau\int_0^{\xi[0,0](t)}\int_{\xi[t,x]^{-1}(0)}^t \mathcal{F}[\rho^*](s,\xi[t,x](s))\cdot\partial_2\xi[t,x](s) \\ &\quad\cdot\left(\partial_1\phi(t,x) + \lambda(t,x)\partial_2\phi(t,x)\right)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\quad+\int_0^\tau\int_0^{\xi[0,0](t)} \frac{u\left(\xi[t,x]^{-1}(0)\right)\partial_2\xi[t,x](\xi[t,x]^{-1}(0))}{\lambda(\xi[t,x]^{-1}(0),0)} \\ &\quad\cdot\left(\partial_1\phi(t,x) + \lambda(t,x)\partial_2\phi(t,x)\right)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Further details are omitted to keep the presentation short.

We refer the readers to [3] for the uniqueness proof of the case without boundary datum. The uniqueness proof here can be derived similarly as in [5], with some adaption due to the existence of boundary datum.

So far, the solution has only been constructed on a sufficiently small time horizon $t^* \in (0,\frac{1}{2L}]$. Clearly, using the semi-group property, we can iterate this to obtain a sequence of initial boundary value problems between time horizons $[0,t^*)$, $[t^*,2t^*),\ldots$ and can exhaust every finite time horizon $T \in \mathbb{R}_{>0}$. ■

Remark 2.2 (Computing $\partial_2\xi$): Recalling the definition of the characteristics in Eq. (II.3), we compute the spatial derivative of the entire equation and end up for $(t,x,\tau) \in \Omega_T \times [0,T]$ with

$$\partial_2\xi[t,x](\tau) = 1 + \int_t^\tau \partial_2\lambda(s,\xi[t,x](\tau))\partial_2\xi[t,x](\tau)\,\mathrm{d}s. \tag{II.7}$$

As $\partial_2\lambda$ is still essentially bounded and the integral equation is linear, we can directly write down the solution as

$$\partial_2\xi[t,x](\tau) = \exp\left(\int_t^\tau \partial_2\lambda(s,\xi[t,x](s))\,\mathrm{d}s\right),$$

from which the positivity of $\partial_2\xi$ then follows.

Remark 2.3 (Well-posedness of the involved mappings):

- Due to the positivity of λ, the characteristics $\xi[t,x]$ at the time-space point $(t,x) \in \Omega_T$ are invertible. The characteristics trace back the solution at $(t,x) \in \Omega_T$ to either boundary or initial datum. The dependency of the solution w.r.t. initial and boundary datum is thus changed along the zero characteristics $\xi[0,0](t)$. However, due to the semi-linearity, there is still a coupling (yet not explicit).
- For the region which is dependent on the initial datum, the mapping $\xi[t,x](0)$ gives the spatial coordinate where the characteristics emanated from at $t=0$ and is thus defined for $x \ge \xi[0,0](t)$, and the mapping $\xi[t,x]^{-1}(0)$ gives back the time where the characteristics emanated from at $x=0$ and is thus defined for $x \le \xi[0,0](t)$.
- The mapping $\partial_2\xi$ comes from the velocity function varying in space, compressing or dispersing the solution in space time to satisfy the conservation/balancing of mass.
- Since the velocity function λ is strictly positive, we obtain also the regularity of the solution $C\left([0,1];L^1((0,T))\right)$. In addition, as we will point out in Corollary 3.1 in a more general setup, the solution is uniformly bounded on Ω_T.

III. IBVP WITH BOUNDARY FEEDBACK CONTROL

In this section, we consider the well-posedness of the system of balance laws subject to a class of boundary controls. By replacing the boundary datum in the model considered in Definition 2.2 by

$$\boldsymbol{\Lambda}(t,0)\boldsymbol{\rho}(t,0) = \boldsymbol{\mathcal{R}}[\boldsymbol{\rho}](t)\odot\boldsymbol{u}(t) \quad t\in[0,T], \tag{III.1}$$

where the feedback boundary control operator $\boldsymbol{\mathcal{R}}$ is to be specified, and the operator $\odot$ is defined as component-wise multiplication of two vectors, resulting into a new vector, i.e., for any two elements $\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$, $\boldsymbol{y}\odot\boldsymbol{z} = \begin{pmatrix} y_1z_1 & y_1z_1 & \ldots & y_nz_n\end{pmatrix}^{\mathrm{T}}$, we then obtain the following coupled IBVP.

Definition 3.1 (The control problem considered): Let Assumption 3.1 hold, the control problem is then stated as

$$\begin{aligned} \boldsymbol{\rho}_t(t,x) + (\boldsymbol{\Lambda}(t,x)\boldsymbol{\rho}(t,x))_x &= \boldsymbol{f}(t,x,\boldsymbol{\rho}(t,x)) && (t,x)\in\Omega_T \\ \boldsymbol{\Lambda}(t,0)\boldsymbol{\rho}(t,0) &= \boldsymbol{\mathcal{R}}[\boldsymbol{\rho}](t)\odot\boldsymbol{u}(t) && t\in(0,T) \\ \boldsymbol{\rho}(0,x) &= \boldsymbol{\rho}_0(x) && x\in(0,1). \end{aligned}$$

We propose the following assumptions.

Assumption 3.1 (Velocity, etc. – vectorial): Let $T \in \mathbb{R}_{>0}$ and assume for the vectorial IBVP in Definition 3.1

- $\boldsymbol{\rho}_0 \in L^\infty\left((0,1);\mathbb{R}^n_{\ge 0}\right)$,
- $\boldsymbol{u} \in L^\infty\left((0,T);\mathbb{R}^n_{\ge 0}\right)$,
- $\boldsymbol{f} \in C(\Omega_T\times\mathbb{R}^n;\mathbb{R}^n):\ \exists L\in\mathbb{R}_{>0},\forall(t,x)\in\Omega_T,$ $\forall \boldsymbol{z},\boldsymbol{y}\in\mathbb{R}^n:\ \|\boldsymbol{f}(t,x,\boldsymbol{z}) - \boldsymbol{f}(t,x,\boldsymbol{y})\| \le L\|\boldsymbol{z}-\boldsymbol{y}\|,$
- $\boldsymbol{\Lambda} \in C\left([0,T];W^{1,\infty}((0,1);\mathbb{R}^{n\times n}_{>0})\right)$ is diagonal.

Since the operator $\boldsymbol{\mathcal{R}}$ is a function of the solution $\boldsymbol{\rho}$ itself, the existence and uniqueness of solutions is not straightforward and has to be proven. This problem is investigated for two different settings, in Section III-A for Lipschitz-continuous (w.r.t. the traffic information $\boldsymbol{\rho}$) operators and in Section III-B for delayed (w.r.t. $\boldsymbol{\rho}$) operators. Please note that the bold notation represents the fact that we now deal with a system of equations and not a single balance law and that we allow the operator $\boldsymbol{\mathcal{R}}$ to be dependent on all those solutions simultaneously. The generality of this dependency is specified in the following section and its reasonability explained in Section IV-A.

A. Lipschitz continuous control operator

We first study well-posedness of the problem in Definition 3.1 subject to Lipschitz-continuous routing operators.

Definition 3.2 (Lipschitz-continuous boundary operator): The boundary operator $\boldsymbol{\mathcal{R}} : C([0,T];L^1(((0,1);\mathbb{R}^n))) \to L^\infty((0,T);[0,1]^n)$ in Eq. (III.1) representing the control impact on the boundary datum in Definition 3.1 is called **Lipschitz-continuous** iff either one of the following holds:

- $\forall \boldsymbol{\rho}^1, \boldsymbol{\rho}^2 \in C([0,T];L^1((0,1);\mathbb{R}^n))$ with $\|\boldsymbol{\rho}^1\|_{L^\infty(\Omega_T)} \leq C, \|\boldsymbol{\rho}^2\|_{L^\infty(\Omega_T)} \leq C,\ C \in \mathbb{R}_{>0},\ \exists L_R(C) \in \mathbb{R}_{>0},\ p \in (1,\infty]$ such that
$$\left\|\mathcal{R}[\boldsymbol{\rho}^1] - \mathcal{R}[\boldsymbol{\rho}^2]\right\|_{L^p((0,t);\mathbb{R}^n)} \leq L_R(C)\|\boldsymbol{\rho}^1 - \boldsymbol{\rho}^2\|_{C([0,t];L^1((0,1);\mathbb{R}^n))}\ \forall t \in [0,T],$$
- $\forall t \in (0,T],\ \forall \boldsymbol{\rho}^1, \boldsymbol{\rho}^2 \in C([0,1];L^1((0,t);\mathbb{R}^n))$ with $\|\boldsymbol{\rho}^1\|_{L^\infty(\Omega_T)} \leq C, \|\boldsymbol{\rho}^2\|_{L^\infty(\Omega_T)} \leq C,\ C \in \mathbb{R}_{>0},$ $\exists L_R(C) \in \mathbb{R}_{>0},\ p \in (1,\infty]$ such that
$$\left\|\mathcal{R}[\boldsymbol{\rho}^1] - \mathcal{R}[\boldsymbol{\rho}^2]\right\|_{L^p((0,t);\mathbb{R}^n)} \leq L_R(C)\|\boldsymbol{\rho}^1 - \boldsymbol{\rho}^2\|_{C([0,1];L^1((0,t);\mathbb{R}^n))}.$$

Remark 3.1 (Lipschitz-continuous operator): The Lipschitz condition in Definition 3.2 satisfies:

- At a given time $t \in [0,T]$, the control operator can depend on the solution between $[0,t]$. This is ensured by the time dependency in the Lipschitz-estimate.
- Due to the general structure of a Lipschitz-estimate in L^p, the control operator is allowed to be as general as possible.
- The case $p = 1$ is excluded since the proof of existence requires a contraction argument which could not (easily) be obtained for $p = 1$. This is because the corresponding Lipschitz constant cannot be made small for small time $t \in (0,T]$, which makes the main idea of proving the well-posedness in the following Theorem 3.1 fail.

1) Well-posedness in the Lipschitz case: The previous assumptions enable us to prove the well-posedness of the resulting IBVP with boundary control.

Theorem 3.1 (Boundary control, Lipschitz): Given Assumption 3.1 the boundary control problem in Definition 3.1 subject to Definition 3.2 admits a unique weak solution $\boldsymbol{\rho} \in C\left([0,T];L^1((0,1);\mathbb{R}^n_{\geq 0})\right)$ in the sense of Definition 2.3 yet in the vectorial setting. In addition, the solution is also the unique solution of the following fixed-point problem in $C\left([0,T];L^1((0,1);\mathbb{R}^n)\right)$ for $(t,x) \in \Omega_T$ a.e.: For $i \in \{1,2,\ldots,n\}$, ρ_i satisfies

$$\rho_i(t,x) = \begin{cases} \int\limits_{\xi_i[t,x]^{-1}(0)}^{t} \mathcal{F}_i[\boldsymbol{\rho}](\tau,\xi_i[t,x](\tau))\partial_2\xi_i[t,x](\tau)\,\mathrm{d}\tau + \\ \frac{u_i\left(\xi_i[t,x]^{-1}(0)\right)\mathcal{R}_i[\boldsymbol{\rho}](\xi_i[t,x]^{-1}(0))\partial_2\xi_i[t,x](\xi_i[t,x]^{-1}(0))}{\lambda_i(\xi_i[t,x]^{-1}(0),0)} & \text{for } x \leq \xi_i[0,0](t) \\ \int\limits_{0}^{t} \mathcal{F}_i[\boldsymbol{\rho}](\tau,\xi_i[t,x](\tau))\partial_2\xi_i[t,x](\tau)\,\mathrm{d}\tau + \\ \rho_{i,0}(\xi_i[t,x](0))\partial_2\xi_i[t,x](0) & \text{for } x > \xi_i[0,0](t) \end{cases} \quad \text{(III.2)}$$

where the characteristics ξ_i and the semi-linearity $\mathcal{F}_i$ are defined similarly as in Theorem 2.1, again in the corresponding vectorial setting.

Proof: This theorem follows for Lipschitz-continuous routing operators in the $C([0,t];L^1((0,1);\mathbb{R}^n))$ topology in a similar way as Theorem 2.1. The difference comes from the coupling through routing operator and semi-linearity. However, since we postulated vectorial Lipschitz continuity for both the routing and the semi-linearity, the estimates are very similar. We do not go into detail but only mention that Eq. (III.2) is a straightforward generalization of Eq. (II.2). Also, for routings in the $C\left([0,1];L^1((0,t);\mathbb{R}^n)\right)$ topology, the result follows analogously. ■

Furthermore, we provide here a proper estimate of the L^∞-norm of the solution with an upper bound, which will be useful in the following analysis.

Corollary 3.1 (Uniform L^∞ estimate of $\boldsymbol{\rho}$): Let Assumption 3.1 hold, the weak solution of Definition 3.1 satisfies for every $i \in \{1,\ldots,n\}$ and for all $t \in [0,T]$

$$\|\rho_i(t,\cdot)\|_{L^\infty((0,1))} \leq C(n)\cdot C_1\Big(t\|\boldsymbol{f}(\cdot,*,\boldsymbol{0})\|_{L^\infty(\Omega_T;\mathbb{R}^n)} + \frac{\|\boldsymbol{u}\|_{L^\infty((0,T);\mathbb{R}^n)}}{\min_{(i,t,x)\in\{1,\ldots,n\}\times\Omega_T}|\lambda_i(t,x)|} + \|\boldsymbol{\rho}_0\|_{L^\infty((0,1);\mathbb{R}^n)}\Big)\mathrm{e}^{LC_1T},$$

with $C_1 := \exp(T\|\partial_2\boldsymbol{\lambda}\|_{L^\infty(\Omega_T;\mathbb{R}^n)})$ and $C(n) \in \mathbb{R}_{\geq 0}$ being a constant dependent only on n.

Proof: The proof uses the fixed-point solution in Eq. (III.2), Assumption 3.1 and Gronwall's lemma. ■

B. Delayed control operator

From the engineering point of view, controls in general are subject to different extents of inevitable delays. In almost every realistic application, a delay in time should be considered to make already existent models precise (for instance, compare [11], [12] for delay in hyperbolic conservation laws). We investigate the well-posedness of the problem in Definition 3.1 subject to a class of delayed control operators specified in Definition 3.3, where "delayed" means that $\boldsymbol{\mathcal{R}}$ at time $t \in [0,T]$ depends only on $\boldsymbol{\rho}$ at previous times.

Definition 3.3 (Delayed boundary control operator): Let a delay $d \in C([0,T];\mathbb{R}_{\geq 0})$ be given and assume that there exists a positive constant $\epsilon \in (0,T)$ such that for $t \in (0,T]$, for any $i \in \{1, \dots, n\}$,

$$\begin{aligned} d(t) &= t \text{ for } t \in [0, \epsilon] \\ 0 < d(t) &< t \text{ for } t \in (\epsilon, T], \end{aligned} \tag{III.3}$$

we call $\boldsymbol{\mathcal{R}} : C\left(([0,T]; L^1((0,1);\mathbb{R}^n)\right) \to L^\infty((0,T);\mathbb{R}^n)$ a routing operator delayed by $d(t), t \in [0,T]$ iff the following holds $\forall t \in [0,T] \forall \boldsymbol{\rho} \in C([0,T]; L^1((0,1);\mathbb{R}^n))$

$$\boldsymbol{\mathcal{R}}[\boldsymbol{\rho}](t) = \boldsymbol{\mathcal{R}}\left[\boldsymbol{\rho}|_{[0,t-d(t)]}\right](t).$$

Remark 3.2 (Regularity in the delayed case): Due to the delay in Definition 3.3, there is no need of any higher regularity on the operator for the system in Definition 3.1 to obtain well-posedness. The introduction of $d(t)$ serves as the prescription of a time-varying delay. The condition $d(0) = 0$ assures that at $t = 0$ the control operator remains well-posed (otherwise, one might need to evaluate $\boldsymbol{\rho}$ at negative times when $\boldsymbol{\rho}$ is not defined). Note that Eq. (III.3) prevents a real-time coupling of the control operator w.r.t. the solution.

Theorem 3.2 (Boundary control, delayed): If Assumption 3.1 and Definition 3.3 hold, the control problem in Definition 3.1 admits a unique weak solution in the sense of Definition 2.3 in its vectorial counterpart.

Proof: Define $\delta := \min\{d(t); t \in [\epsilon, T]\} > 0$, the existence and positiveness of δ comes directly from the continuity and positiveness assumption on d.

Step One. From the assumption that $d(t)$ satisfies, the left boundary control operator on the time interval $[0, \epsilon]$ is $\Lambda(t,0)\boldsymbol{\rho}(t,0) = \boldsymbol{\mathcal{R}}[\boldsymbol{\rho}_0](t)$, which can be considered as pre-defined at time $t = 0$ and thus does not rely on any solution history of the IBVP in Definition 3.1. The solution of the IBVP in Definition 3.1 can then be derived following the proof of Theorem 2.1 yet in the vectorial case. That is, for $t \in [0, \epsilon]$, the solution is also the unique solution in $C\left([0,\epsilon]; (L^1((0,1);\mathbb{R}^n)\right)$ for $(t,x) \in \Omega_T$ a.e. of the fixed-point problem Eq. (II.2) yet in the vectorial case, with $\boldsymbol{u}\left(\boldsymbol{\xi}[t,x]^{-1}(0)\right)$ replaced by $\boldsymbol{\mathcal{R}}[\boldsymbol{\rho}_0]\left(\boldsymbol{\xi}[t,x]^{-1}(0)\right) \odot \boldsymbol{u}\left(\boldsymbol{\xi}[t,x]^{-1}(0)\right)$.

Step Two. For $t \geq \epsilon$, the induction method can be applied for the derivation of the solution. For $t \in [\epsilon, \epsilon + \delta]$ with δ as given before, from which $t - d(t) \in [0, \epsilon + \delta - d(t)] \subset [0, \epsilon]$, the solution of the IBVP in Definition 3.1 can be derived as follows: for any $i \in \{1, \dots, n\}$,

$$\rho_i(t,x) = \begin{cases} \int_{\xi_i[t,x]^{-1}(0)}^{t} \mathcal{F}[\boldsymbol{\rho}](\tau, \xi_i[t,x](\tau)) \partial_2 \xi_i[t,x](\tau)\,\mathrm{d}\tau \\ + \frac{u_i\left(\xi_i[t,x]^{-1}(0)\right)}{\lambda_i(\xi_i[t,x]^{-1}(0),0)} \mathcal{R}_i[\boldsymbol{\rho}|_{[0,\epsilon]}](\xi_i[t,x]^{-1}(0)) \\ \cdot \partial_2 \xi_i[t,x](\xi_i[t,x]^{-1}(0)) \qquad \text{for } x \leq \xi_i[0,0](t), \\ \int_0^t \mathcal{F}[\boldsymbol{\rho}](\tau, \xi_i[t,x](\tau)) \partial_2 \xi_i[t,x](\tau)\,\mathrm{d}\tau \\ + \rho_{i,0}(\xi_i[t,x](0)) \partial_2 \xi_i[t,x](0) \text{ for } x > \xi_i[0,0](t). \end{cases}$$

Assume that the solution on the time interval $[0, \epsilon + n\delta], n \in \mathbb{N}_{>0}$ is known, then, on the time interval $[\epsilon + n\delta, \epsilon + (n+1)\delta]$, we have $t - d(t) \in [0, \epsilon + n\delta]$ and the solution can be obtained similarly. By induction, the solution on the whole time interval of interest $[0, T]$ can be obtained.

By following the proof of Theorem 2.1, the function constructed as above is indeed a weak solution to Definition 3.1 subject to Definition 2.3 and Definition 3.3. ■

IV. INSTANTIATIONS

The routing choices describe how the flows are allocated from one link to another at a given junction within the networked traffic system (see Fig. 1). We propose a time-dependent in-coming flow $\boldsymbol{u}(t)$ and then, a routing operator, also named as boundary controller, will decide the distribution of this inflow onto each of the out-going links. Clearly, for effective traffic management, one would like to choose routing based on the traffic density $\rho_i, i \in \{1, \cdots, n\}$ of all the out-going links or in the network case even based on the density on all possible routes in the network.

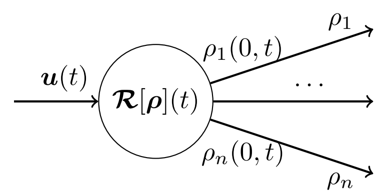

Fig. 1. Illustration of the archetype of a traffic network.

For simplicity, we consider in this work only one intersection, keeping in mind that the entire theory also holds for networked systems where the routing at a specific intersection can also depend on traffic density on the network links not adjacent to the intersection.

A. Lipschitz continuous control operator

We propose the following Lipschitz routing operators.

Example 4.1 (Routing inspired by the Logit-routing): Inspired by [13], define for $i \in \{1, \dots, n\}$ that

$$\mathcal{R}_i[\boldsymbol{\rho}](t) := \frac{\mathrm{e}^{-\rho_i(t,0)}}{\sum_{i=1}^n \mathrm{e}^{-\rho_i(t,0)}}, \quad t \in [0,T],$$

which comes from an intuitive requirement of the routing. Indeed, it is reasonable to assign more inflow to the link with less traffic density, and visa versa.

Next, we study whether $\boldsymbol{\mathcal{R}} : C([0,T]; (L^1(0,1))^n) \to L^\infty((0,T);\mathbb{R}^n_{\geq 0})$ satisfies Definition 3.2. To this end, let $i \in \{1, 2, \dots, n\}$ and take $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2 \in C([0,T]; L^1((0,1);\mathbb{R}^n))$, then for any $t \in [0,T]$:

$$\begin{aligned} &\left\|\mathcal{R}_i[\boldsymbol{\rho}^1] - \mathcal{R}_i[\boldsymbol{\rho}^2]\right\|^2_{L^2(0,t)} \\ &\leq \int_0^t \left(\frac{\mathrm{e}^{-\rho_i^1(\tau,0)}}{\sum_{i=1}^n \mathrm{e}^{-\rho_i^1(\tau,0)}} - \frac{\mathrm{e}^{-\rho_i^2(\tau,0)}}{\sum_{i=1}^n \mathrm{e}^{-\rho_i^2(\tau,0)}} \right)^2 \mathrm{d}\tau \\ &= \int_0^t \left(\frac{\mathrm{e}^{-\rho_i^1(\tau,0)} \sum_{i=1}^n \mathrm{e}^{-\rho_i^2(\tau,0)} - \mathrm{e}^{-\rho_i^2(\tau,0)} \sum_{i=1}^n \mathrm{e}^{-\rho_i^1(\tau,0)}}{\left(\sum_{i=1}^n \mathrm{e}^{-\rho_i^1(\tau,0)}\right)\left(\sum_{i=1}^n \mathrm{e}^{-\rho_i^2(\tau,0)}\right)} \right)^2 \mathrm{d}\tau. \end{aligned}$$

Due to the nonnegativity of the density, we can estimate for any $\tau \in [0,t]$ that

$$\left| \mathrm{e}^{-\rho_i^1(\tau,0)} \sum_{i=1}^n \mathrm{e}^{-\rho_i^2(\tau,0)} - \mathrm{e}^{-\rho_i^2(\tau,0)} \sum_{i=1}^n \mathrm{e}^{-\rho_i^1(\tau,0)} \right|$$

$$= \Big| \mathrm{e}^{-\rho_i^1(\tau,0)} \sum_{j=1}^n \left(\mathrm{e}^{-\rho_i^2(\tau,0)} - \mathrm{e}^{-\rho_j^1(\tau,0)} \right) + \left(\mathrm{e}^{-\rho_i^1(\tau,0)} - \mathrm{e}^{-\rho_i^2(\tau,0)} \right) \sum_{j=1}^n \mathrm{e}^{-\rho_j^1(\tau,0)} \Big|$$
$$\leq (1+n) \sum_{i=1}^n |\rho_i^2(\tau,0) - \rho_i^1(\tau,0)|,$$

where the intermediate value theorem is applied. From Corollary 3.1 and Hölder's inequality we obtain for $i \in \{1, \dots, n\}$

$$\begin{aligned}
&\left\| \mathcal{R}_i[\boldsymbol{\rho}^1] - \mathcal{R}_i[\boldsymbol{\rho}^2] \right\|_{L^2(0,t)}^2 \\
&\leq \int_0^t \left(\frac{(1+n) \sum_{i=1}^n |\rho_i^2(\tau,0) - \rho_i^1(\tau,0)|}{\left(\sum_{i=1}^n \mathrm{e}^{-\rho_i^1(\tau,0)}\right)\left(\sum_{i=1}^n \mathrm{e}^{-\rho_i^2(\tau,0)}\right)} \right)^2 \mathrm{d}\tau \\
&\leq \int_0^t \left(\frac{(1+n) \sum_{i=1}^n |\rho_i^2(\tau,0) - \rho_i^1(\tau,0)|}{\mathrm{e}^{-\|\rho_i^1(\tau,\cdot)\|_{L^\infty(0,1)}} \mathrm{e}^{-\|\rho_i^2(\tau,\cdot)\|_{L^\infty(0,1)}}} \right)^2 \mathrm{d}\tau \\
&\leq (1+n)^2 C^2 \sum_{i=1}^n \int_0^t \left(\rho_i^2(\tau,0) - \rho_i^1(\tau,0) \right)^2 \mathrm{d}\tau \\
&= (1+n)^2 C^2 \|\boldsymbol{\rho}^1(\cdot,0) - \boldsymbol{\rho}^2(\cdot,0)\|_{L^2((0,t);\mathbb{R}^n)}^2 \\
&\leq L_1^2 \|\boldsymbol{\rho}^1 - \boldsymbol{\rho}^2\|_{C([0,1];L^1((0,t);\mathbb{R}^n)))}^2, \qquad \text{(IV.1)}
\end{aligned}$$

with $C := \exp\left(\|\boldsymbol{\rho}^1\|_{L^\infty(\Omega_T)} + \|\boldsymbol{\rho}^2\|_{L^\infty(\Omega_T)}\right)$ and $L_1 := (1+n)C$. From the above estimate, it can be further derived that $\boldsymbol{\mathcal{R}}$ satisfies Definition 3.2 as a Lipschitz-continuous routing operator.

B. Delayed control operator

Example 4.2 (Minimal-density routing with delay): Let $\epsilon \in \mathbb{R}_{>0}$ and for $s, t \in \mathbb{R}_{\geq 0}$,

$$d(t) := \epsilon \operatorname{sat}\left(\tfrac{t}{\epsilon}\right) := \begin{cases} t & t \in [0,\epsilon], \\ \epsilon & t > \epsilon. \end{cases} \qquad \text{(IV.2)}$$

Define

$$S[\boldsymbol{\rho}](t) := \underset{i \in \{1,\dots,n\}}{\arg\min} \int_0^1 \rho_i(t,x)\,\mathrm{d}x, \ t \geq 0$$

and let $|\cdot|$ denote the cardinality of a set. Define an operator

$$\begin{aligned}
&\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \cdots, \mathcal{R}_n)^{\mathrm{T}} \\
&: C([0,T]; L^1((0,1);\mathbb{R}^n)) \to L^\infty((0,T);[0,1]^n),
\end{aligned}$$

where for any $t \in [0,T]$,

$$\mathcal{R}_i[\boldsymbol{\rho}](t) = \begin{cases} \frac{1}{|S[\boldsymbol{\rho}](t-d(t))|}, & \text{for } i \in S[\boldsymbol{\rho}](t) \\ 0, & \text{for } i \notin S[\boldsymbol{\rho}](t) \end{cases}$$

with d as in Eq. (IV.2). This delayed routing operator satisfies Definition 3.3. Following Theorem 3.2, the IBVP as defined in Definition 3.1 subject to this boundary routing operator $\boldsymbol{\mathcal{R}}$ admits a unique weak solution.

Note that this routing operator is generally not even continuous with respect to the density, however, when applying a delay as done in this instantiation, the system is well-posed.

V. Future Works

As we are more interested in the routing, we have kept the link dynamics as simple as possible and focus on the properties the routing has to satisfy to obtain a well-posed system. For example, the link dynamics considered in this work are rather trivial in that the spill-back or shock behaviors which are classical and natural for traffic flow models are not incorporated. Consideration of these behaviors in our model will need to be taken into account in the future. Another future work worth considering is the stability analysis of the IBVPs w.r.t. input parameter. Even though this might be straightforward for the Lipschitz-continuous routing operators, it needs to be studied carefully in the case of delayed routing operators in the proper topology. Moreover, as we have exploited the necessary assumptions on the involved routing operators, these can be generalized to more sophisticated PDE models in traffic flow as the LWR model [14], [15].

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